New Results on Minimal Strongly Imperfect Graphs

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Abstract

The characterization of strongly perfect graphs by a restricted list of forbidden induced subgraphs has remained an open question for a long time. The minimal strongly imperfect graphs which are simultaneous imperfect are only odd holes and odd antiholes (E. Olaru, [9]), but the entire list is not known, in spite of a lot of particular results in this direction. In this paper we give some new properties of the minimal strongly imperfect graphs. Thus we introduce the notion of critical (co-critical) pair of vertices, and we prove that any vertex of a minimal s-imperfect (minimal c-imperfect) graph is contained in a critical (co-critical) pair, and, in a minimal s-imperfect graph different from a cycle of length at least 5, any vertex is the center of a star cutset, or, if not, it belongs to a house or a domino. Also, we characterize the triangle-free minimal s-imperfect graphs. (By s-perfect we mean the complement of a strongly perfect graph.)

1 Introduction

Throughout this paper all graphs $G = (V, E)$ are finite, undirected, and simple (i.e. without loops and multiple edges).

A *hole* is an induced cycle of length at least five, whereas an *antihole* is the complement of a hole. By P_k, C_k we denote an induced path on $k \geq 3$ vertices, and an induced cycle on $k \geq 3$ vertices, respectively.

For $A \subseteq V$, $G(A)$ is the subgraph of G induced by the set A; shortly, by *subgraph* we mean an *induced subgraph*, when no confusion arises.

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By (A, B) we mean the set $\{ab : a \in A, b \in B, ab \in E\}$, where $A, B \subset V, A \cap B = \emptyset$, and we write $A \sim B$ whenever $ab \in E$ holds for all $a \in A$ and $b \in B$: if $(A, B) = \emptyset$, we write $A \not\sim B$. For $\{x\} \sim A$ (or $\{x\} \not\sim A$) we briefly note $x \sim A$ ($x \not\sim A$, respectively). A vertex set $A \subset V(G)$ is *homogeneous* in G if for any vertex x in $V(G) - A$, we have or $x \sim A$ or $x \not\sim A$; a *module* is a homogeneous set with at least two elements.

The neighborhood of a vertex x in a graph G (i.e. the set $\{y : y \in V(G), x \sim y\}$) will be denoted by $N(x)$, and $N(x) \cup \{x\}$ by $N[x]$. For $G - N[x]$ we note $\overline{N}(x)$.

An *independent set* in G is a set of mutually non-adjacent vertices. A *stable set*[3] in G is a maximal (with respect to set inclusion) independent set, and the *stability number* $\alpha(G)$ of G is the maximum cardinality of a stable set. By $S(G)$ we denote the family of all stable sets of G, and $\mathcal{S}_\alpha(G) = \{S : S \in \mathcal{S}(G), |S| = \alpha(G)\}$. A *complete set* in G is a subset A of vertices pairwise adjacent. A *clique* [4] in G is a maximal complete set in G. By $\mathcal{C}(G)$ we denote the family of all cliques of G, and the *clique number* $\omega(G)$ is the maximum cardinality of a clique. Clearly, $\mathcal{C}(G) = \mathcal{S}(\overline{G}), \mathcal{C}_\omega(G) = \mathcal{S}_\alpha(\overline{G}), \mathcal{S}_\alpha(G) \subseteq \mathcal{S}(G)$ and $\mathcal{C}_\omega(G) \subseteq \mathcal{C}(G)$ holds for any graph G.

The *chromatic number* and the *clique covering number* of G are, respectively, $\chi(G)$ and $\theta(G)$, where $\theta(G) = \chi(\overline{G})$.

A graph G is *perfect* if $\alpha(H) = \theta(H)$ (or, equivalently, by Perfect Graph Theorem, $\chi(H) = \omega(H)$) holds for any subgraph H of G.

A graph G is *minimal imperfect* if it is not perfect, but every induced subgraph of G is perfect.

Let M be a set, $F = \{M_i\}_{i \in I}$ a family of subsets of M and T a subset of M. The set T is called a transversal of F iff $T \cap M_i \neq \emptyset$ for all $i \in I$. A transversal T of F is called perfect iff $|T \cap M_i| = 1$ for all $i \in I$.

It is easy to check that for any graph G, a transversal T of the family $\mathcal{S}(G)$ of stable sets (or of the family $\mathcal{C}(G)$ of cliques) is perfect iff T is a clique (stable set) of G.

Therefore a perfect transversal of $\mathcal{S}(G)$, which is a clique will be called a s-transversal, and, similarly, a perfect transversal of $\mathcal{C}(G)$ will be called a c-transversal. We will use also the name of stable transversal and clique transversal.

A perfect transversal of G is a c-transversal or a s-transversal, according

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[3]In the literature the notions "stable set" and "independent set" sometimes have the same meaning. We prefer to use them as seen above, and as in papers such as [2, 10, 13].

[4]Idem for the notions "clique" and "complete set".

to context.

Clearly, because $\mathcal{C}(G) = \mathcal{S}(\overline{G})$ and $\mathcal{S}(G) = \mathcal{C}(\overline{G})$, any c-transversal of G is a s-transversal of the complement $\overline{G}$, and conversely.

Different classes of graphs may be defined by the notion of perfect transversal of some families of cliques and/or stable sets. Thus, the perfect graphs can be defined in terms of perfect transversals as:

> *A graph G is perfect iff for any induced subgraph H, the family $\mathcal{S}_\alpha(H)(\text{or } \mathcal{C}_\omega(H))$ has a perfect transversal.*

An interesting subclass of perfect graphs is obtained by using the perfect transversals for the family of all cliques or all stable sets of any induced subgraphs as follows.

Definition 1 *A graph G is called:*

- *c-perfect iff for each of its induced subgraphs H, the family $\mathcal{C}(H)$ has a c-transversal;*

- *s-perfect iff for each of its induced subgraphs H, the family $\mathcal{S}(H)$ has a s-transversal;*

- *(c,s)-perfect iff for each of its induced subgraphs H, the family $\mathcal{S}(H)$ has a s-transversal, and the family $\mathcal{C}(H)$ has a c-transversal.*

More precisely, a graph G is *c-perfect* iff for any induced subgraph H, there is a stable set $S \in \mathcal{S}(H)$, so that for any clique $Q \in \mathcal{C}(H)$, $Q \cap S \neq \varnothing$ holds; similarly, a graph G is *s-perfect* iff for each of its induced subgraphs H, there is a clique $Q \in \mathcal{C}(H)$, such that $Q \cap S \neq \varnothing$, for any stable set of H.

Definition 2 *A graph G is called*

- *minimal c-imperfect, if it is not c-perfect, but any proper induced subgraph is c-perfect;*

- *minimal s-imperfect, if it is not s-perfect, but any proper induced subgraph is s-perfect;*

- *minimal (c,s)-imperfect, if it is not (c,s)-perfect, but any proper induced subgraph is (c,s)-perfect.*

A graph G is *c-perfect* if and only if $\overline{G}$ is s-perfect; a graph G is (c,s)-perfect if and only if G and $\overline{G}$ are *c-perfect* (or s-perfect). A graph G is minimal *c-imperfect* if and only if $\overline{G}$ is minimal s-imperfect.

So, by using the complement, we can easily switch between *c-perfect* and *s-perfect*, but to broach the structure problems, it is convenient to use the separate terminology.

Note that c-perfect graphs are also known under the name of *strongly perfect graphs* and are introduced by C. Berge and P. Duchet [1]. In the recent paper [7] of Hougardy, the *s-perfect* graphs appear under the name of *co-strongly perfect graphs*.

We will use the common name of *strongly perfect* for any graph which is *c-perfect, s-perfect, or (c,s)-perfect*, and also *minimal strongly imperfect* for any graph in the classes given by Definition 2.

Let us notice that the *strongly perfect* graphs form an interesting class of perfect graphs with many practical applications .(For example, they serve as one of the best mathematical models for a real situation where one would like to choose an optimal set of leaders from a given set of people (party, club, etc.))[13].

Also, the class of strongly perfect graphs contains some subclasses (e.g. comparability, parity, Meyniel) for which there are efficient algorithms to recognize and to solve optimization problems. We consider that it is possible to find polynomial algorithms of low degree (possible linear) to recognize the class of strongly perfect graphs and to solve optimization problems on it.

In order to recognize this class, a way is to find the minimal forbidden subgraphs. In this paper we give some new properties of the minimal strongly imperfect graphs.

2 Minimal Strongly Imperfect Graphs

Let us remark that a graph G is c-imperfect (s-imperfect, (c,s)-imperfect) iff it contains a minimal c-imperfect (s-imperfect, (c,s)-imperfect) as induced subgraph (i.e. there is an induced c-imperfect subgraph H such that for any vertex $x \in V(H)$, the subgraph $H - x$ is c-perfect (s-perfect, (c,s)-perfect)).

Therefore, only the minimal strongly imperfect graphs are forbidden for the strong perfection.

Although for perfection the minimal forbidden subgraphs are only the holes and antiholes (*Strong perfect graph theorem*, conjectured by C.Berge in 1960, and proved in 2002 by M. Chudnovsky, N. Robertson, P. Seymour,

R. Thomas [4]), the problem of finding the list of minimal forbidden induced subgraphs for the class of strongly perfect graphs has remained open for a long time in spite of several particular results in this direction. Let us remark that the minimal strongly imperfect graphs which are, simultaneously, imperfect are completely characterized by E.Olaru in 1993 [10],[9]. He proved that these are precisely the holes of odd length ≥ 5, and their complements. To characterize the strongly perfect graphs by forbidden induced subgraphs it is helpful to first find new properties of perfect minimal strongly imperfect graphs.
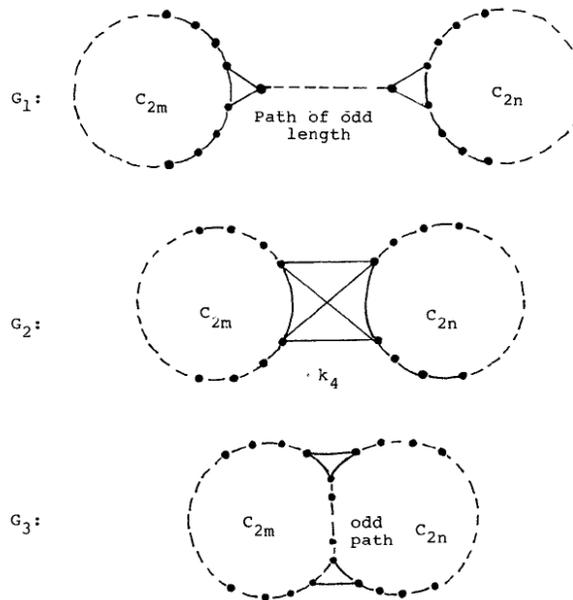

Figure 1:

There are known some classes of minimal c-imperfect and perfect graphs given by Ravindra (see Figure 1) [13], Berge (see Figure 2), Chvatal (see Figure 3), and Maffray (see Figure 4).

Proposition 1 *Any minimal strongly imperfect graph is connected.*

Proof: Let G be a minimal c-imperfect graph. We suppose that G is not connected. Then, every connected component is c-perfect, and for each of them there is a stable set which meet all of its cliques. The union of these stable sets is a stable set of G, and meets all the cliques in G, a contradiction.
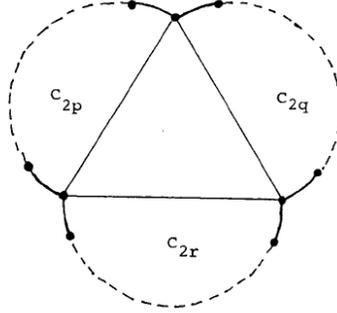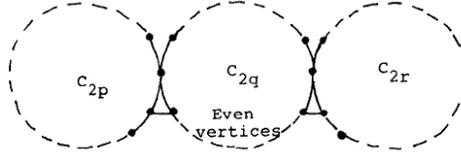
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Figure 2:

Figure 3:

Let G be a minimal s-imperfect graph. We suppose that G is not connected. Then, every connected component is s-perfect, and for each of them there is a clique which meets all of its stable sets. Let C be a connected component and Q the clique that meets all the stable sets of C. Because any stable set of G is a union of stable sets, one in each component, we have that Q meets any stable set of G, a contradiction.

In the following we introduce the notion of critical pair, which generalizes the important notion of α -critical edge.

Definition 3 *The pair (x, y) of vertices of a graph G is called critical in G if there are S_x, S_y, stable sets in G, containing the vertex x, respectively y, so that $S_x - \{x\} \subset S_y$.*

The pair (x, y) is called co-critical, if it is critical in $\overline{G}$ (i.e. there are Q_x, Q_y, cliques in G, containing the vertex x, respectively y, so that $Q_x - \{x\} \subset Q_y$).

We remark that if the pair of vertices (x, y) is critical, then $xy \in E(G)$, otherwise, if $xy \notin E(G)$, then $S_x \cup \{y\}$ is an independent set in G,
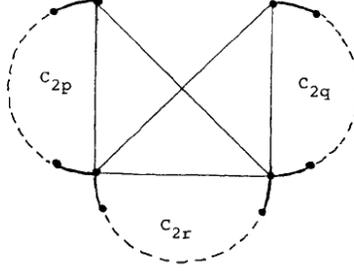
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Figure 4:

a contradiction with the maximality of S_x. From this fact, if the pair of vertices (x, y) is critical, we call the edge xy *critical edge*. Analogously, if the pair of vertices (x, y) is co-critical, then xy is a edge in the complement of G, and we call it a *co-critical edge* in G.

Also,if we restrict the family $\mathcal{S}(G)$ to $\mathcal{S}_\alpha(G)$, then any critical pair in G is an α-*critical edge* .

Proposition 2 *In a minimal s-imperfect graph, any vertex belongs to a critical pair.*

Proof: Let G be a minimal s-imperfect graph and x a vertex of G. Because G is a minimal s-imperfect graph, it follows that $G - x$ is s-perfect. Then there is a clique $Q \in \mathcal{C}(G - x)$, with $Q \cap S \neq \varnothing$, for any stable set $S \in \mathcal{S}(G - x)$. We remark that $Q \in \mathcal{C}(G)$, because otherwise, $Q \cup \{x\}$ is a clique in G which intersect all the stable sets of G, in contradiction with the fact that G is a minimal s-imperfect graph. Then there is $S_x \in \mathcal{S}(G)$ so that $Q \cap S_x = \varnothing$. Because $S_x - \{x\}$ is an independent set in $G - x$, it follows that there is a stable set $S' \in \mathcal{S}(G - x)$, with $S_x - \{x\} \subset S'$. We have also $Q \cap S' \neq \varnothing$. Let be $\{y\} = Q \cap S'$. We observe that $y \sim x$, otherwise, $S_x \cup \{y\}$ is an independent set in G, in contradiction with the maximality of S_x. We have than $S' \in \mathcal{S}(G)$. Because $y \in S'$, we can denote S' by S_y. So, (x, y) is a critical pair.

Proposition 3 *In a minimal c-imperfect graph, any vertex belongs to a co-critical pair.*

Proof: Let G be a minimal c-imperfect graph and $x \in V(G)$. Then $\overline{G}$ is minimal s-imperfect, and from the Proposition 2 it follows that in $\overline{G}$, x

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is incident with a critical edge in x, which is equivalent with the fact that there is a vertex y such that the pair (x, y) is co-critical in G.

Theorem 1 *A graph is s-perfect if any induced subgraph has a vertex which is not incident with any critical edges.*

Proof: Let G be a graph with the property that any induced subgraph H has a vertex which is not incident with any critical edge. Let us show that G is s-perfect. We suppose that G is not s-perfect. Then there is an induced subgraph H, minimal s-imperfect. From the Proposition 2 we have that any vertex x is incident with a critical edge in x, a contradiction.

Definition 4 *A vertex of a graph is called semi-simplicial if it is not midpoint for any induced path of length three.*

Remark 1 *This definition is equivalent to the following (see E. Olaru,[10], Theorem 3):Let G be a graph, $x \in V(G)$, and let $\overline{G}(N_1), \overline{G}(N_2), ..., \overline{G}(N_p)$ be the connected components of $\overline{G}(N(x))$. The vertex x is called semi-simplicial if all the sets N_1, N_2, ..., N_p are homogeneous sets in G.*

We recall that a homogeneous set in the graph G is a vertex set $A \subset V(G)$ with the property that for any vertex x in $V(G) - A$, we have either $x \sim A$, or $x \nsim A$.

Proposition 4 *([10]) If G is a minimal s- imperfect or minimal c-imperfect graph then neither G nor its complement $\overline{G}$ has a semi-simplicial vertex.*

A graph G is called *brittle* if every induced subgraph H of G contains a vertex that is not an endpoint nor a midpoint of a P_4 in H,(i.e. H and $\overline{H}$ has no semi-simplicial vertices).

Corollary 1 *Every brittle graph is (c,s)-perfect.*

For a set A which is homogeneous in G, let G/A denote the graph which is obtained from G by identifying all vertices of A (i.e. we obtain G/A from G by replacing the set A with a new vertex a and connecting a by edges with all the vertices $x \in G - A$ satisfying $x \sim A$).

Proposition 5 *([10]) Let G be a graph, A a homogeneous set in G so that $G(A)$ has a perfect transversal. Then G has a perfect transversal if and only if G/A has a perfect transversal.*
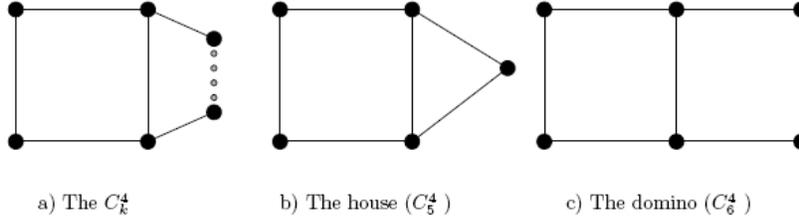
a) The C_k^4 b) The house (C_5^4)   c) The domino (C_6^4)

Figure 5:

Corollary 2 *No minimal strongly imperfect graph has a module.*

Proof: Let G be a minimal c-imperfect (minimal s-imperfect) graph and A a homogeneous set in G. We suppose that $|A| \geq 2$. Then $G(A)$ and G/A are *c-perfect* (s-perfect) graphs, because they are proper induced subgraphs, and they have perfect transversals. By Proposition 5 we have that G has a perfect transversal, which yields a contradiction.

Proposition 6 *If G is a minimal s-imperfect graph different from a hole, then for every vertex x of G, one of the next conditions holds:*
    1. x is the center of a star cutset;
    2. x belongs to a house or a domino.

Proof: That is a corollary of the following

Theorem 2 *([11]) Let G be a graph and x a vertex of G. Then one of the next statements holds:*
    1. x is a semi-simplicial vertex;
    2. x is the center of a star cutset;
    3. $x \in C_k^4$, $k \geq 5$.

Here C_k^4 denote an induced cycle C_k, $k \geq 5$, or a cycle of length at least 5, with only a chord forming a C_4 and a $C_{k-2}, k \geq 5$ (see Figure 5).
    By $x \in C_k^4$ we mean that x is a vertex of the hole C_k (in the first case), or a vertex of C_4 which is not an extremity of the chord (in the second case).
    Let G be a minimal s-imperfect graph. Then , by Proposition 4 , G has no semi-simplicial vertices. It follows, by Theorem 2, that any vertex x is either the center of a star cutset, or $x \in C_k^4$, $k \geq 5$, and by hypothesis C_k^4

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has no hole. Suppose that x is no center of a star cut set. It follows that $x \in C_5^4$, a house $(\overline{P}_5)$ or $x \in C_6^4$, a domino. (For $k \geq 7$, C_k^4 contains a C_{k-2} (a hole)).

3   Triangle-Free Minimal S-Imperfect Graphs

A triangle-free graph is a graph without cycles of length three. We will prove that triangle-free graphs are s-perfect if and only if they are hole-free (i.e. $C_k - free, \; k \geq 5$).

Definition 5 *Let us call an edge semi-simplicial if it is not interior of an induced path with 4 vertices.*

First we give the following result.

Proposition 7 *A clique with two vertices of a graph G is a s-transversal if and only if the edge induced by its vertices is semi-simplicial.*

Proof: Let $Q = \{a, b\}$ be a clique in G with two vertices, such that there is no P_4 containing in interior both vertices a and b. We prove that Q is a s-transversal in G. We suppose that Q is not a s-transversal. Then there is a stable set S, with $Q \cap S = \varnothing$. Because S is a stable set, $a, b \notin S$,and Q is a clique, there are vertices $x, y \in S$, with $x \sim a, y \sim b$, $x \neq y, x \nsim b, y \nsim a$. It follows that a and b are interior of $P_4 = [x, a, b, y]$, a contradiction.

Conversely, let $Q = \{a, b\}$ be a clique with two vertices, which is a s-transversal, and suppose that there is an induced path of length 3, containing a and b in interior, say $P_4 = [x, a, b, y]$. Then, for any stable set S, containing the vertices x, y , $S \cap Q = \varnothing$ holds, contradicting the definition of s-transversal.

Proposition 8 *A triangle free graph G is minimal s-imperfect if and only if $G = C_k, k \geq 5$.*

Proof: If $G = C_k$,$k \geq 5$, then G is obviously triangle-free and minimal *s-imperfect*. Conversely, in minimal s-imperfect graph triangle free, any edge is not semi-simplicial, because the extremities of any edge induce a clique in G , which has no s-transversal. From a result due to V. Chvatal and I. Rusu [6] (" If a graph is $\mathbf{C}_k$-free, $k \geq 5$, then it has an edge which is not interior of a P_4."), we deduce that G contains a hole C_k. Because G is minimal s-imperfect, it must be $G = C_k$.

Corollary 3 *A triangle free graph is s-perfect if and only if it is hole -free (i.e. C_k-free, $k \geq 5$).*

Proof: It follows from Proposition 8.

Corollary 4 *A bipartite graph G is minimal s-imperfect if and only if $G = C_{2k}, k \geq 3$.*

Proof: If the graph is bipartite, then it is triangle free, and the result follows from Proposition 8.

Corollary 5 *A bipartite graph G is s-perfect if and only if it contains no even hole (i.e.G is $C_{2k} - free, k \geq 3$). Equivalently, the complement of a bipartite graph is strongly perfect (i.e. c-perfect) if and only if it contains no even antihole.*

Remark 2 *Because any bipartite graph is c-perfect, we have: A bipartite graph is (c,s)-perfect if and only if it is hole-free.*

Acknowledgments

References

[1] C. Berge, P. Douchet, *Strongly perfect graphs*, Annals of Discrete Mathematics, 21, pp. 57-61, 1984.

[2] F. Bonomo, M. Chudnovsky, G. Duran, *Partial characterizations of clique-perfect graphs I: subclasses of claw-free graphs*, Discrete Applied Mathematics, 156(7), pp. 1058-1082, 2008.

[3] M.Chudnovsky, G.Cornuéjols, X. Liu, P. Seymour, K. Vušković, *Recognizing Berge graphs*, Combinatorica, 25, pp. 143-186, 2005.

[4] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, *The strong perfect graph theorem*, Annals of Mathematics, 164, pp. 51-229, 2006.

[5] V. Chvatal, *Perfectly orderable graphs*, in: Topics on perfect graphs, Annals of Discrete Mathematics, 21, pp. 63-65, 1984.

[6] V.Chvatal, I.Rusu, *A Note on Graphs Without Long Holes*, 1st Workshop on perfect graphs, Princeton, New Jersey, U.S.A., 1993.

[7] S. Hougardy, *Classes of perfect graphs*, Discrete Mathematics, 306 (19-20), pp. 2529-2571, 2006.

[8] B. Jamison, S. Olariu, *On the semi-perfect elimination*, Advances in Applied Mathematics, 9, pp. 364-376, 1988.

[9] E. Olaru, *The structure of imperfect critically strongly-imperfect graphs*, Discrete Mathematics, 156, pp. 299-302, 1996.

[10] E. Olaru, *On strongly perfect graphs and the structure of critically-imperfect graphs*, Analele Ştiinţifice ale Universităţii "Al.I.Cuza" Iaşi, Tomul II, Informatică, pp. 45-59, 1993.

[11] E. Olaru, G. Alexe, *On the graph structure of k-transitive symmetric relations*, Studii şi Cercetări Matematice (Mathematical Reports), Tomul 50, 5-6, pp. 395-407, 1998.

[12] G. Ravindra, *Meyniel graphs are strongly perfect*, Journal of Combinatorial Theory, Series B, 35(2), pp. 187-190, 1982.

[13] G. Ravindra, *Some classes of strongly perfect graphs*, Discrete Mathematics, 206, pp. 197-203, 1999.