Boolean Lifting Properties
for Bounded Distributive Lattices

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Abstract

In this paper, we introduce the lifting properties for the Boolean elements of bounded distributive lattices with respect to the congruences, filters and ideals, we establish how they relate to each other and to significant algebraic properties, and we determine important classes of bounded distributive lattices which satisfy these lifting properties.

Keywords: (Boolean) Lifting Property, (B–normal, B–conormal, Filt–local, Id–local) bounded distributive lattice, Boolean center, congruence, filter, ideal, radical

1 Introduction

Several kinds of lifting properties have been studied in rings and in residuated structures. In ring theory, the Lifting Idempotents Property (LIP) ([21], [1], [6], [16]), which is the property that the idempotent elements can be lifted modulo every left (respectively right) ideal of a (unitary) ring, is closely related to clean rings and exchange rings (in the commutative case, rings with LIP coincide to clean rings and to exchange rings). It is well known that the idempotent elements of a commutative unitary ring R form a Boolean algebra, called the *Boolean center* of R. Properties similar to LIP have been studied in other algebras which have a Boolean center, namely residuated structures: MV–algebras ([10]), BL–algebras ([9], [17]) and residuated lattices ([11], [18], [12], [7]); the property studied in these

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algebras refers to the lifting of the Boolean elements modulo filters, and was called the Boolean Lifting Property (BLP). In [7], we have introduced a Boolean Lifting Property (BLP) modulo filters for bounded distributive lattices, we have studied the BLP for residuated lattices from the algebraic and the topological point of view, and we have proved that the BLP is transferrable, through the reticulation functor, between the class of residuated lattices and the class of bounded distributive lattices. This transfer through the reticulation, as well as the multitude of algebraic and topological properties the BLP is related to, has motivated us to pursue the study of the BLP for bounded distributive lattices, which we have initiated with the present article. Besides the properties which are similar in this case to those which appear in residuated lattices and their subclasses, due to the transfer we have mentioned, there are some differences between the situation that occurs in bounded distributive lattices and the one we had met in residuated structures, the first of which is the fact that the naturally occurring Boolean Lifting Properties with respect to filters and to ideals appear as particular cases of the BLP for congruences, so we have had to define a general notion of a BLP for an arbitrary class of congruences, from which all other Boolean Lifting Properties derive. Remarkable classes of bounded distributive lattices have turned out to have either the strongest version of the BLP, namely the one for all the congruences, or the BLP for filters or ideals: Boolean algebras, chains and arbitrary direct products of chains, B–normal, B–conormal, Filt–local and Id–local bounded distributive lattices. We intend to continue the research in this paper, both by extending this study for bounded distributive lattices and by applying the results we obtain here to residuated lattices and other algebras of non–classical logics. Furthermore, the study of the BLP for bounded distributive lattices, along with the possibility of transfer of the BLP through the reticulation, might allow us to connect the work concerning the BLP in the algebras of logic to that regarding the LIP in commutative rings.

The present paper is structured as follows: the section of preliminaries below precedes two sections made of original work, with the only exception of the results cited from other papers and those provided with proofs, but pointed out as being previously known. In Section 3, we introduce the different Boolean Lifting Properties, provide algebraic characterizations for them and give examples of classes of bounded distributive lattices in which they are present. In Section 4, we prove that arbitrary direct products preserve the BLP, that the BLP for filters is equivalent to B–conormality,

and, when the Boolean center is trivial, but the lattice in question is not, also to Filt–locality; dually, the BLP for ideals is equivalent to B–normality, and, when the Boolean center is trivial and the bounded distributive lattice is non–trivial, also to Id–locality; these properties enable us to provide further examples of classes of bounded distributive lattices which satisfy the different kinds of BLP, and classes whose members do not satisfy these kinds of BLP.

2 Preliminaries

In this section we recall some definitions, notations and properties related to bounded distributive lattices that we shall use in the sequel. We refer the reader to [2], [3], [4], [13], [22] for a further study of the notions and results presented here. For the sake of completeness, we shall provide proofs for some of these results.

Let $\mathbb{N}$ be set of the natural numbers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For any set M, we shall denote by $\Delta_M = \{(x,x) \mid x \in M\}$, and by $\nabla_M = M^2$. Throughout this paper, whenever there is no danger of confusion, algebraic structures will be designated by their underlying sets. Everywhere in this paper, unless mentioned otherwise, we shall be using these classical notations for the operations of a (bounded) lattice or a Boolean algebra: if L is a lattice, or a bounded lattice, or a Boolean algebra, then we denote the algebraic structure of L by $(L, \wedge, \vee)$, $(L, \wedge, \vee, 0, 1)$ or $(L, \wedge, \vee, \neg, 0, 1)$, respectively. Also, unless mentioned otherwise, the partial order of any of these kinds of structures will be denoted by $\leq$. If L is a poset or a bounded poset, then its algebraic structure will be denoted by $(L, \leq)$ or $(L, \leq, 0, 1)$, respectively.

It is well known that complete lattices are bounded and, in particular, finite lattices are bounded. A bounded lattice is said to be *trivial* iff it has only one element, that is it has $0 = 1$, and it is said to be *non–trivial* iff it has $0 \neq 1$. It is well known that, if $(L, \leq)$ is a chain, then $(L, \min, \max)$ is a distributive lattice.

The dual of a poset $(L, \leq)$ or a bounded poset $(L, \leq, 0, 1)$ is the poset $(L, \geq)$, respectively the bounded poset $(L, \geq, 1, 0)$. The dual of a lattice $(L, \wedge, \vee)$, a bounded lattice $(L, \wedge, \vee, 0, 1)$ or a Boolean algebra $(L, \wedge, \vee, \neg, 0, 1)$ is the lattice $(L, \vee, \wedge)$, the bounded lattice $(L, \vee, \wedge, 1, 0)$ or the Boolean algebra $(L, \vee, \wedge, \neg, 1, 0)$, respectively. Notice that, unlike the notions of a $\wedge$–semilattice and a $\vee$–semilattice, each of the notions of a lattice, a distributive lattice, a bounded lattice and a Boolean algebra is dual to itself.

Trivially, a surjective lattice morphism between two bounded lattices

is a bounded lattice morphism. If L_1 and L_2 are lattices, then a function $h : L_1 \to L_2$ is called a *lattice anti–morphism* iff h is a lattice morphism between L_1 and the dual of L_2. h is called a *lattice anti–isomorphism* iff h is a lattice isomorphism between L_1 and the dual of L_2; in this case, the lattices L_1 and L_2 are said to be *anti–isomorphic*. The same goes if L_1 and L_2 are bounded lattices, respectively Boolean algebras.

Throughout the rest of this section, unless mentioned otherwise, L will be an arbitrary bounded distributive lattice.

The *Boolean center* of L will be denoted by $\mathcal{B}(L)$; the elements of $\mathcal{B}(L)$ are called *Boolean elements* of L. We recall that $\mathcal{B}(L)$ is, by definition, the set of the complemented elements of L, and that $\mathcal{B}(L)$ is a bounded sublattice of L, and thus a Boolean algebra. Trivially, L is a Boolean algebra iff $\mathcal{B}(L) = L$. Just as for any Boolean algebra, the complement of any $e \in \mathcal{B}(L)$ will be denoted by $\neg e$. It is straightforward that, for all $x \in L$ and all $e \in \mathcal{B}(L)$: $x \vee e = 1$ iff $x \geq \neg e$, and $x \wedge e = 0$ iff $x \leq \neg e$. Clearly, if L is a bounded chain, then $\mathcal{B}(L) = \{0, 1\}$.

For any bounded distributive lattices L_1 and L_2 and every bounded lattice morphism $f : L_1 \to L_2$, the inclusion $f(\mathcal{B}(L_1)) \subseteq \mathcal{B}(L_2)$ holds. Hence we can define $\mathcal{B}(f) : \mathcal{B}(L_1) \to \mathcal{B}(L_2)$, for all $x \in \mathcal{B}(L_1)$, $\mathcal{B}(f)(x) = f(x)$. Then, obviously, $\mathcal{B}(f)$ is a Boolean morphism, and $\mathcal{B}$ becomes a covariant functor from the category of bounded distributive lattices to the category of Boolean algebras. Clearly, this also holds for bounded lattice anti–morphisms, since the notion of complement is dual to itself and thus the notion of Boolean center is dual to itself. If $h : L_1 \to L_2$ is a bounded lattice anti–morphism, then $\mathcal{B}(h)$ is a Boolean anti–morphism.

A non–empty subset F of L is called a *filter* of L iff, for any $x, y \in L$:
if $x, y \in F$, then $x \wedge y \in F$;
if $x \in F$ and $x \leq y$, then $y \in F$.

A non–empty subset I of L is called an *ideal* of L iff, for any $x, y \in L$:
if $x, y \in I$, then $x \vee y \in I$;
if $x \in I$ and $x \geq y$, then $y \in I$.

Clearly, the notion of ideal is dual to that of filter, that is the ideals of L are the filters of the dual of L, and the filters of L are the ideals of the dual of L.

The set of the filters of L will be denoted by $\mathrm{Filt}(L)$, and the set of the ideals of L will be denoted by $\mathrm{Id}(L)$. $\{1\}$ is the smallest element of $\mathrm{Filt}(L)$, and L is the largest element of $\mathrm{Filt}(L)$, with respect to set inclusion. $\{1\}$ is called the *trivial filter*, and L is called the *improper filter* of L. Any filter

having at least two different elements is called a *non–trivial filter*, and any filter which is a proper subset of L is called a *proper filter* of L. Clearly, a filter of L is proper iff it does not contain 0. Dually, $\{0\}$ is the smallest element of $\mathrm{Id}(L)$, and L is the largest element of $\mathrm{Id}(L)$, with respect to set inclusion. $\{0\}$ is called the *trivial ideal*, and L is called the *improper ideal* of L. Any ideal having at least two different elements is called a *non–trivial ideal*, and any ideal which is a proper subset of L is called a *proper ideal* of L. An ideal of L is proper iff it does not contain 1. Clearly, $\mathrm{Filt}(L) \cap \mathrm{Id}(L) = \{L\}$.

A *congruence* of L is an equivalence on L which preserves $\wedge$ and $\vee$, that is an equivalence $\sim$ on the set L with the property that, for all $x, y, x', y' \in L$ such that $x \sim x'$ and $y \sim y'$, it follows that $x \wedge x' \sim y \wedge y'$ and $x \vee x' \sim y \vee y'$. The set of the congruences of L will be denoted by $\mathrm{Con}(L)$. Δ_L is the smallest element of $\mathrm{Con}(L)$, and ∇_L is the largest element of $\mathrm{Con}(L)$, with respect to set inclusion. Clearly, the notion of *congruence* is dual to itself, that is the congruences of L coincide to the congruences of the dual of L.

The intersection of any family of filters of L is a filter of L. Hence, for every $X \subseteq L$, there exists the smallest filter of L (with respect to $\subseteq$) which includes X; this filter is denoted by $[X)$ and called the *filter of L generated by X*. For every $a \in L$, $[\{a\})$ is denoted, simply, by $[a)$, and called the *principal filter generated by a*. It is immediate that $[\emptyset) = \{1\}$ and, for all $\emptyset \neq X \subseteq L$, $[X) = \{x \in L \mid (\exists\, n \in \mathbb{N}^*)\,(\exists\, x_1, \ldots, x_n \in X)\,(x_1 \wedge \ldots \wedge x_n \leq x)\}$. Hence, for all $a \in L$, $[a) = \{x \in L \mid a \leq x\}$; thus: $[a) = \{1\}$ iff $a = 1$, and $[a) = L$ iff $a = 0$. Notice, additionally, that $[\emptyset) = [1)$ and, for all $n \in \mathbb{N}^*$ and all $x_1, \ldots, x_n \in L$, $[\{x_1, \ldots, x_n\}) = [x_1 \wedge \ldots \wedge x_n)$; so every finitely generated filter (that is filter generated by a finite subset of L) is a principal filter; in particular, if L is finite, then $\mathrm{Filt}(L) = \mathrm{PFilt}(L)$.

Dually, the same goes for ideals. For every $X \subseteq L$, the *ideal generated by X* is denoted by $(X]$, and, for every $a \in L$, the *principal ideal generated by a* is denoted by $(a]$. $(\emptyset] = \{0\}$ and, for all $\emptyset \neq X \subseteq L$, $(X] = \{x \in L \mid (\exists\, n \in \mathbb{N}^*)\,(\exists\, x_1, \ldots, x_n \in X)\,(x_1 \vee \ldots \vee x_n \geq x)\}$. For all $a \in L$, $(a] = \{x \in L \mid a \geq x\}$; $(a] = \{0\}$ iff $a = 0$, and $(a] = L$ iff $a = 1$. $(\emptyset] = (0]$ and, for all $n \in \mathbb{N}^*$ and all $x_1, \ldots, x_n \in L$, $(\{x_1, \ldots, x_n\}] = (x_1 \vee \ldots \vee x_n]$; so every finitely generated ideal is a principal ideal; in particular, if L is finite, then $\mathrm{Id}(L) = \mathrm{PId}(L)$.

This also holds for congruences: the intersection of any family of congruences of L is a congruence of L. For every $Y \subseteq L^2$, the smallest congruence of L which includes Y is denoted by $Cg(Y)$ and called the

congruence of L generated by Y. For any $a, b \in L$, $Cg(\{(a,b)\})$ is denoted, simply, by $Cg(a,b)$, and called the *principal congruence of L generated by* $(a,b) \in L^2$; according to [4], for all $x, y \in L$, $(x,y) \in Cg(a,b)$ iff $x \vee a \vee b = y \vee a \vee b$ and $x \wedge a \wedge b = y \wedge a \wedge b$. Clearly, for all $a \in L$, $Cg(a,a) = \Delta_L$, because each congruence is reflexive. Whenever the lattice L needs to be specified, for all $Y \subseteq L^2$ and all $a, b \in L$, we shall denote $Cg_L(Y)$ and $Cg_L(a,b)$ instead of $Cg(Y)$ and $Cg(a,b)$, respectively.

If F and G are filters of L, then we shall denote by $F \vee G = [F \cup G)$. It is straightforward that, for any $F, G \in \mathrm{Filt}(L)$, $F \vee G = \{a \in L \mid (\exists\, x \in F)\,(\exists\, y \in G)\,(x \wedge y \leq a)\}$. More generally, if $(F_t)_{t \in T}$ is a family of filters of L, then we denote by $\bigvee_{t \in T} F_t = [\bigcup_{t \in T} F_t)$. Dually, if $I, J \in \mathrm{Id}(L)$, then we denote $I \vee J = (I \cup J]$. For any $I, J \in \mathrm{Id}(L)$, $I \vee J = \{a \in L \mid (\exists\, x \in I)\,(\exists\, y \in J)\,(x \vee y \geq a)\}$. If $(I_t)_{t \in T} \subseteq \mathrm{Id}(L)$, then we denote $\bigvee_{t \in T} I_t = (\bigcup_{t \in T} I_t]$. The same goes for congruences: if $\sim, \equiv\, \in \mathrm{Con}(L)$, then we denote $\sim \vee \equiv\, = Cg(\sim \cup \equiv)$, and, if $(\theta_t)_{t \in T} \subseteq \mathrm{Con}(L)$, then we denote $\bigvee_{t \in T} \theta_t = Cg(\bigcup_{t \in T} \theta_t)$. With the operations defined as above, $(\mathrm{Filt}(L), \cap, \vee, \{1\}, L)$, $(\mathrm{Id}(L), \cap, \vee, \{0\}, L)$ and $(\mathrm{Con}(L), \cap, \vee, \Delta_L, \nabla_L)$ become bounded distributive lattices, with partial order $\subseteq$; moreover, each of them is a complete lattice. We shall denote by $\mathrm{PFilt}(L)$ the set of the principal filters of L, and by $\mathrm{PId}(L)$ the set of the principal ideals of L. Clearly, for all $a, b \in L$: $[a) \vee [b) = [a \wedge b)$, $[a) \cap [b) = [a \vee b)$, $(a] \vee (b] = (a \vee b]$, $(a] \cap (b] = (a \wedge b]$ and we have seen, above, that $[1) = \{1\}$, $[0) = L$, $(0] = \{0\}$ and $(1] = L$, hence $\mathrm{PFilt}(L)$ is a bounded sublattice of $\mathrm{Filt}(L)$ and $\mathrm{PId}(L)$ is a bounded sublattice of $\mathrm{Id}(L)$. Moreover, L is anti–isomorphic to $\mathrm{PFilt}(L)$, and isomorphic to $\mathrm{PId}(L)$; indeed, if we define $f : L \to \mathrm{PFilt}(L)$ and $g : L \to \mathrm{PId}(L)$ by: for all $a \in L$, $f(a) = [a)$ and $g(a) = (a]$, then f is a bounded lattice anti–isomorphism, and g is a bounded lattice isomorphism. In particular, the bounded (distributive) lattices $\mathrm{PFilt}(L)$ and $\mathrm{PId}(L)$ are anti–isomorphic; this is not the case for $\mathrm{Filt}(L)$ and $\mathrm{Id}(L)$, which are not even always in bijection (take, for instance, the bounded distributive lattice $(\mathbb{N}, \gcd, \mathrm{lcm}, \mid, 1, 0)$ of the set of natural numbers ordered by the relation "divides"; it can be easily verified that this lattice has all filters principal, which means that its set of filters is countable, while its set of ideals in is bijection to the set of the subsets of $\mathbb{N}$). Furthermore, if we apply the functor $\mathcal{B}$ to f and g, we get the Boolean anti–isomorphism $\mathcal{B}(f) : \mathcal{B}(L) \to \mathcal{B}(\mathrm{PFilt}(L))$ and the Boolean isomorphism $\mathcal{B}(g) : \mathcal{B}(L) \to \mathcal{B}(\mathrm{PId}(L))$; in particular, $\mathcal{B}(f)$ and $\mathcal{B}(g)$ are

bijections, hence $\mathcal{B}(\mathrm{PFilt}(L)) = \mathcal{B}(f)(\mathcal{B}(L)) = f(\mathcal{B}(L)) = \{[e) \mid e \in \mathcal{B}(L)\}$ and $\mathcal{B}(\mathrm{PId}(L)) = \mathcal{B}(g)(\mathcal{B}(L)) = g(\mathcal{B}(L)) = \{(e] \mid e \in \mathcal{B}(L)\}$. Clearly, for all $a \in L$, $([a), \wedge, \vee, a, 1)$ and $((a], \wedge, \vee, 0, a)$ are bounded distributive lattices and sublattices of L. It is straightforward and well known that, for all $e \in \mathcal{B}(L)$, the bounded distributive lattice L is isomorphic to each of the direct products $[e) \times [\neg e)$ and $(e] \times (\neg e]$.

Let L_1 and L_2 be bounded distributive lattices and $f : L_1 \to L_2$ be a bounded lattice morphism. Then, for every $G \in \mathrm{Filt}(L_2)$, $f^{-1}(G) \in \mathrm{Filt}(L_1)$, and, dually, for every $J \in \mathrm{Id}(L_2)$, $f^{-1}(J) \in \mathrm{Id}(L_1)$. If f is surjective, then we also have: for every $F \in \mathrm{Filt}(L_1)$, $f(F) \in \mathrm{Filt}(L_2)$, and, dually, for every $I \in \mathrm{Id}(L_1)$, $f(I) \in \mathrm{Id}(L_2)$.

A proper filter P of L is called a *prime filter* iff, for all $x, y \in L$, if $x \vee y \in P$, then $x \in P$ or $y \in P$. Dually, a proper ideal Q of L is called a *prime ideal* iff, for all $x, y \in L$, if $x \wedge y \in Q$, then $x \in Q$ or $y \in Q$. We shall denote the set of the prime filters of L by $\mathrm{Spec}_{\mathrm{Filt}}(L)$, and the set of the prime ideals of L by $\mathrm{Spec}_{\mathrm{Id}}(L)$.

A maximal element of the set of proper filters of L (ordered by $\subseteq$) is called a *maximal filter*. Dually, a maximal element of the set of proper ideals of L is called a *maximal ideal*. We shall denote the set of the maximal filters of L by $\mathrm{Max}_{\mathrm{Filt}}(L)$, and the set of the maximal ideals of L by $\mathrm{Max}_{\mathrm{Id}}(L)$. It is well known that any maximal filter of L is a prime filter of L and, dually, any maximal ideal is a prime ideal. Thus: $\mathrm{Max}_{\mathrm{Filt}}(L) \subseteq \mathrm{Spec}_{\mathrm{Filt}}(L) \subseteq \mathrm{Filt}(L) \setminus \{L\}$ and $\mathrm{Max}_{\mathrm{Id}}(L) \subseteq \mathrm{Spec}_{\mathrm{Id}}(L) \subseteq \mathrm{Id}(L) \setminus \{L\}$. It is an immediate consequence of Zorn's Lemma that any non–trivial bounded distributive lattice has maximal filters and maximal ideals and, moreover, if L is non–trivial, then any proper filter of L is included in a maximal filter and any proper ideal of L is included in a maximal ideal. Thus, obviously, L has maximal filters iff L has proper filters iff $\{1\}$ is a proper filter of L iff L is non–trivial iff $\{0\}$ is a proper ideal of L iff L has proper ideals iff L has maximal ideals.

Proposition 1 [4]

1. *For every $a, b \in L$, $Cg(a, b) \in \mathcal{B}(\mathrm{Con}(L))$. If $a \leq b$, then $\neg\, Cg(a, b) = Cg(0, a) \vee Cg(b, 1)$.*

2. $\mathcal{B}(\mathrm{Con}(L)) = \{\bigvee_{i=1}^{n} Cg(a_i, b_i) \mid n \in \mathbb{N}^*, (\forall\, i \in \overline{1, n})\, (a_i, b_i \in L)\}.$

3. $\mathrm{Con}(L)$ *is a Boolean algebra iff L is finite.*

It is well known that, to every filter F of L, we can associate a congruence of L which we shall denote $\sim_F$, defined by: for all $x, y \in L$, $(x, y) \in \sim_F$ iff $x \wedge a = y \wedge a$ for some $a \in F$. Dually, to every ideal I of L, we can associate a congruence $\sim_I$ of L, defined by: for all $x, y \in L$, $(x, y) \in \sim_I$ iff $x \vee a = y \vee a$ for some $a \in I$. These notations pose no danger of confusion, because, as we have seen, the only subset of L which is both a filter and an ideal of L is L, and, clearly, the congruence $\sim_L$ associated to the filter L coincides to the congruence $\sim_L$ associated to the ideal L, namely $\sim_L = L^2 = \nabla_L$. It is immediate that, for every $a, x, y \in L$: $(x, y) \in \sim_{[a)}$ iff $x \wedge a = y \wedge a$; $(x, y) \in \sim_{(a]}$ iff $x \vee a = y \vee a$. Hence $\sim_{\{1\}} = \sim_{[1)} = \Delta_L = \sim_{(0]} = \sim_{\{0\}}$.

It is straightforward that, if $(L_t)_{t \in T}$ is a non–empty family of bounded distributive lattices and $L = \prod_{t \in T} L_t$, then $\mathcal{B}(L) = \prod_{t \in T} \mathcal{B}(L_t) = \{(e_t)_{t \in T} \mid (\forall\, t \in T)\,(e_t \in \mathcal{B}(L_t))\}$, $\mathrm{Filt}(L) = \{\prod_{t \in T} F_t \mid (\forall\, t \in T)\,(F_t \in \mathrm{Filt}(L_t))\}$, $\mathrm{Id}(L) = \{\prod_{t \in T} I_t \mid (\forall\, t \in T)\,(I_t \in \mathrm{Id}(L_t))\}$ and $\mathrm{Con}(L) = \{\prod_{t \in T} \theta_t \mid (\forall\, t \in T)\,(\theta_t \in \mathrm{Con}(L_t))\}$, where we have denoted by $\prod_{t \in T} \theta_t = \{((a_t)_{t \in T}, (b_t)_{t \in T}) \mid (\forall\, t \in T)\,((a_t, b_t) \in \theta_t)\}$ for every family $(\theta_t)_{t \in T}$ with $\theta_t \in \mathrm{Con}(L_t)$ for all $t \in T$. Notice that, for any family $(F_t)_{t \in T}$ with $F_t \in \mathrm{Filt}(L_t)$ for all $t \in T$, we have: $\sim_{\prod_{t \in T} F_t} = \prod_{t \in T} \sim_{F_t}$; similarly, for any family $(I_t)_{t \in T}$ with $I_t \in \mathrm{Id}(L_t)$ for all $t \in T$, we have: $\sim_{\prod_{t \in T} I_t} = \prod_{t \in T} \sim_{I_t}$.

For every $\theta \in \mathrm{Con}(L)$ and any $a \in L$, we shall denote by a/θ the congruence class of a with respect to θ. Also, for any $X \subseteq L$, we shall denote $X/\theta = \{a/\theta \mid a \in X\}$. Then $L/\theta = \{a/\theta \mid a \in L\}$ becomes a bounded distributive lattice, with the operations defined canonically. L/θ is called the *quotient bounded (distributive) lattice of L with respect to θ*. The canonical surjection $p_\theta : L \to L/\theta$, for all $a \in L$, $p_\theta(a) = a/\theta$, is a bounded lattice morphism. Thus p_θ is order–preserving, which means that, for all $x, y \in L$, if $x \leq y$, then $p_\theta(x) \leq p_\theta(y)$, that is $x/\theta \leq y/\theta$. Hence, if L is a chain, then so is L/θ. It is straightforward that $\mathrm{Con}(L/\theta) = \{\equiv /\theta \mid \; \equiv \in \mathrm{Con}(L), \theta \subseteq \equiv\}$, where, for every $\equiv \in \mathrm{Con}(L)$ such that $\theta \subseteq \equiv$, we denote by $\equiv /\theta = \{(a/\theta, b/\theta) \mid a, b \in L, (a, b) \in \equiv\}$.

Notice that a lattice congruence (that is a congruence defined as above) on a Boolean algebra is a Boolean algebra congruence; in other words, if L is

a Boolean algebra and θ is an equivalence on L which preserves $\wedge$ and $\vee$, then θ also preserves $\neg$. Indeed, let L be a Boolean algebra and θ be a (lattice) congruence on L; then, obviously, L/θ is a bounded distributive lattice; also, for every $x \in L$, $x/\theta \wedge (\neg x)/\theta = (x \wedge \neg x)/\theta = 0/\theta$ and $x/\theta \vee (\neg x)/\theta = (x \vee \neg x)/\theta = 1/\theta$; hence L/θ is also complemented, thus it is a Boolean algebra, and, for every $x \in L$, $\neg (x/\theta) = (\neg x)/\theta$; hence, if $x, y \in L$ such that $(x, y) \in \theta$, that is $x/\theta = y/\theta$, then $(\neg x)/\theta = \neg (x/\theta) = \neg (y/\theta) = (\neg y)/\theta$, hence $(\neg x, \neg y) \in \theta$, so θ preserves $\neg$.

If $F \in \mathrm{Filt}(L)$, then we shall denote: for all $a \in L$, by $a/F = a/\sim_F$; for all $X \subseteq L$, by $X/F = X/\sim_F$; thus $L/F = L/\sim_F$, called the *quotient bounded (distributive) lattice of L with respect to F*; by $p_F = p_{\sim_F} : L \to L/F$ the canonical surjective bounded lattice morphism. If L is a chain, then L/F is a chain. Clearly, $1/F = F$ (F is one of the congruence classes of $\sim_F$). It is straightforward that $\mathrm{Filt}(L/F) = \{G/F \mid G \in \mathrm{Filt}(L), F \subseteq G\}$ and $\mathrm{Max}_{\mathrm{Filt}}(L/F) = \{M/F \mid M \in \mathrm{Max}_{\mathrm{Filt}}(L), F \subseteq M\}$.

Similarly, if $I \in \mathrm{Id}(L)$, then we shall denote: for all $a \in L$, by $a/I = a/\sim_I$; for all $X \subseteq L$, by $X/I = X/\sim_I$; thus $L/I = L/\sim_I$, called the *quotient bounded (distributive) lattice of L with respect to I*; by $p_I = p_{\sim_I} : L \to L/I$ the canonical surjective bounded lattice morphism. If L is a chain, then L/I is a chain. $0/I = I$ (I is one of the congruence classes of $\sim_I$). We have: $\mathrm{Id}(L/I) = \{J/I \mid J \in \mathrm{Id}(L), I \subseteq J\}$ and $\mathrm{Max}_{\mathrm{Id}}(L/I) = \{N/I \mid N \in \mathrm{Max}_{\mathrm{Id}}(L), I \subseteq N\}$.

For every $\theta \in \mathrm{Con}(L)$ and each $X \subseteq L$, we denote by $X/\theta = p_\theta(X) = \{x/\theta \mid x \in X\}$. Also, for every $F \in \mathrm{Filt}(L)$, every $I \in \mathrm{Id}(L)$ and each $X \subseteq L$, we denote by $X/F = \{x/F \mid x \in X\}$ and by $X/I = \{x/I \mid x \in X\}$.

For every $a \in L$ and each $\theta \in \mathrm{Con}(L)$, we have: $[a)/\theta = [a/\theta)$ and, dually, $(a]/\theta = (a/\theta]$; indeed, clearly $[a)/\theta \subseteq [a/\theta)$, while, if $b \in L$ such that $b/\theta \in [a/\theta)$, then $a/\theta \leq b/\theta$, thus $b/\theta = a/\theta \vee b/\theta = (a \vee b)/\theta \in [a)/\theta$ since $a \vee b \in [a)$. Consequently: for every $a \in L$, each $F \in \mathrm{Filt}(L)$ and each $I \in \mathrm{Id}(L)$, we have: $[a)/F = [a/F)$, $(a]/F = (a/F]$, $(a]/I = (a/I]$ and $[a)/I = [a/I)$.

Clearly, for any $\theta \in \mathrm{Con}(L)$ and any $e \in \mathcal{B}(L)$, we have: $e/\theta \in \mathcal{B}(L/\theta)$, and $\neg (e/\theta) = \neg e/\theta$ in the Boolean algebra $\mathcal{B}(L/\theta)$. This is because $e/\theta \vee \neg e/\theta = (e \vee \neg e)/\theta = 1/\theta$ and $e/\theta \wedge \neg e/\theta = (e \wedge \neg e)/\theta = 0/\theta$. Hence $\mathcal{B}(L)/\theta \subseteq \mathcal{B}(L/\theta)$.

We have the mappings $F \to \sim_F$ from $\mathrm{Filt}(L)$ to $\mathrm{Con}(L)$ and $I \to \sim_I$ from $\mathrm{Id}(L)$ to $\mathrm{Con}(L)$; these mappings are injective, because, as mentioned above, if $F \in \mathrm{Filt}(L)$, then $1/\sim_F = 1/F = F$, and, dually, if $I \in \mathrm{Id}(L)$,

then $0/\sim_I = 0/I = I$. Also, to every congruence of L, we can associate a filter and an ideal of L: if $\theta \in \mathrm{Con}(L)$, then it is immediate that $1/\theta \in \mathrm{Filt}(L)$ and $0/\theta \in \mathrm{Id}(L)$. Notice that none of these mappings is necessarily bijective. For instance, let $L = \{0, a, 1\}$ be the three–element chain, with $0 < a < 1$; then L is a bounded distributive lattice. Since L is finite, we have $\mathrm{Filt}(L) = \mathrm{PFilt}(L) = \{\{1\}, [a), L\}$ and $\mathrm{Id}(L) = \mathrm{PId}(L) = \{\{0\}, (a], L\}$. But the equivalences corresponding to the following four different partitions of L are congruences of L: $P_1 = \{\{0\}, \{a\}, \{1\}\}$, $P_2 = \{\{0, a\}, \{1\}\}$, $P_3 = \{\{0\}, \{a, 1\}\}$, $P_4 = \{L\}$; indeed, if we denote, for every $i \in \overline{1,4}$, by θ_i the equivalence which corresponds to P_i, then $\theta_1 = \Delta_L$, $\theta_2 = \sim_{(a]}$, $\theta_3 = \sim_{[a)}$ and $\theta_4 = \nabla_L$, so $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathrm{Con}(L)$, hence the cardinality of $\mathrm{Con}(L)$ is strictly greater that of $\mathrm{Filt}(L)$ and that of $\mathrm{Id}(L)$, thus $\mathrm{Con}(L)$ is not in bijection to $\mathrm{Filt}(L)$, nor to $\mathrm{Id}(L)$. Furthermore, see in Example 6 a congruence which does not correspond to any filter, nor to any ideal, that is an element of $\mathrm{Con}(L) \setminus (\{\sim_F \mid F \in \mathrm{Filt}(L)\} \cup \{\sim_I \mid I \in \mathrm{Id}(L)\})$. However, it is well known that, in the particular case when L is a Boolean algebra, the sets $\mathrm{Con}(L)$, $\mathrm{Filt}(L)$ and $\mathrm{Id}(L)$ are in bijection (actually pairwise isomorphic or anti–isomorphic as bounded distributive lattices), because the mapping $\theta \to 1/\theta$ from $\mathrm{Con}(L)$ to $\mathrm{Filt}(L)$ is the inverse of the mapping $F \to \sim_F$ from $\mathrm{Filt}(L)$ to $\mathrm{Con}(L)$, and the mapping $\theta \to 0/\theta$ from $\mathrm{Con}(L)$ to $\mathrm{Id}(L)$ is the inverse of the mapping $I \to \sim_I$ from $\mathrm{Id}(L)$ to $\mathrm{Con}(L)$ (see above the fact that, in this particular case, $\mathrm{Con}(L)$ is exactly the set of the Boolean algebra congruences of L).

3   Boolean Lifting Properties

In this section, we introduce the different kinds of Boolean Lifting Properties which appear in bounded distributive lattices, we study their behaviour with respect to quotients and inverse images through surjective morphisms, and we provide several examples of classes of bounded distributive lattices in which these properties are present, as well as some concrete examples which show how these different Boolean Lifting Properties relate to each other.
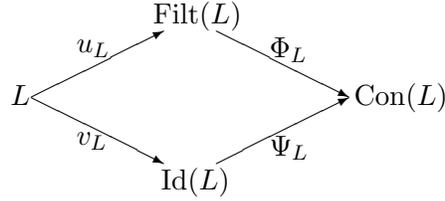
Throughout this section, unless mentioned otherwise, L will be an arbitrary bounded distributive lattice.

Remark 1 *Since the Boolean center of any bounded chain is $\{0, 1\}$, it follows that any Boolean algebra with more than three elements is not a chain, and any chain with at least three elements is not a Boolean algebra.*

    Moreover, if $(L_t)_{t \in T}$ is a non–empty family of bounded chains and $L = \prod_{t \in T} L_t$, then, since $\mathcal{B}(L) = \prod_{t \in T} \mathcal{B}(L_t)$, the following hold:

(a) if L_k has cardinality at least 3 for some $k \in T$, then L is not a Boolean algebra;

(b) if at least two of the bounded chains in the family $(L_t)_{t \in T}$ are non–trivial, then L is not a chain;

(c) consequently, if, for some $k, j \in T$ such that $k \neq j$, L_k has cardinality at least 3 and L_j is non–trivial, then L is neither a chain, nor a Boolean algebra.

    Let us define the following functions:

$$\begin{array}{ccc}
 & \mathrm{Filt}(L) & \\
\nearrow^{u_L} & & \searrow^{\Phi_L} \\
L & & \mathrm{Con}(L) \\
\searrow_{v_L} & & \nearrow_{\Psi_L} \\
 & \mathrm{Id}(L) &
\end{array}$$

- for all $a \in L$, $u_L(a) = [a)$ and $v_L(a) = (a]$;

- for all $F \in \mathrm{Filt}(L)$, $\Phi_L(F) = \sim_F$;

- for all $F \in \mathrm{Id}(L)$, $\Psi_L(I) = \sim_I$.

    Clearly, u_L is an injective bounded lattice anti–morphism and v_L is an injective bounded lattice morphism. It is straightforward that Φ_L and Ψ_L are injective bounded lattice morphisms; as pointed out in Section 2, their injectivity follows from the fact that, for every filter F and any ideal I of L, $1/F = F$ and $0/I = I$; in the particular case when L is a Boolean algebra, Φ and Ψ are bounded lattice isomorphisms.

Remark 2 *The form of the principal congruences and that of $\sim_{[a)}$ and $\sim_{(a]}$, along with Proposition 1, 1, show that, for all $a \in L$:*

- $\Phi_L(u_L(a)) = \sim_{[a)} = Cg(a, 1) \in \mathcal{B}(\mathrm{Con}(L))$;

- $\Psi_L(v_L(a)) = \sim_{(a]} = Cg(0, a) \in \mathcal{B}(\mathrm{Con}(L))$;

- $\Phi_L(u_L(a)) = \neg \Psi_L(v_L(a))$

Lemma 1 *For every $F, G \in \mathrm{Filt}(L)$ such that $F \subseteq G$, we have: $\sim_F \subseteq \sim_G$ and $\sim_{G/F} = \sim_G / \sim_F$.*

Dually, for every $I, J \in \mathrm{Id}(L)$ such that $I \subseteq J$, we have: $\sim_I \subseteq \sim_J$ and $\sim_{J/I} = \sim_J / \sim_I$.

Proof: Let F and G be filters of L such that $F \subseteq G$. Then, since Φ_L is a lattice morphism, it follows that $\Phi(F) \subseteq \Phi(G)$, that is $\sim_F \subseteq \sim_G$. Thus there exist $\sim_{G/F}, \sim_G / \sim_F \in \mathrm{Con}(L/F)$. $\sim_{G/F} = \{(x/F, y/F) \mid x, y \in L, (\exists a \in G)\,(x/F \wedge a/F = y/F \wedge a/F)\}$; $\sim_G / \sim_F = \{(x/\sim_F, y/\sim_F) \mid x, y \in L, (x, y) \in \sim_G\} = \{(x/F, y/F) \mid x, y \in L, (x, y) \in \sim_G\} = \{(x/F, y/F) \mid x, y \in L, (\exists a \in G)\,(x \wedge a = y \wedge a)\} \subseteq \{(x/F, y/F) \mid x, y \in L, (\exists a \in G)\,(x/F \wedge a/F = y/F \wedge a/F)\} = \sim_{G/F}$. Now let $x, y \in L$ such that $x/F, y/F \in \sim_{G/F}$, that is there exists an $a \in G$ such that $x/F \wedge a/F = y/F \wedge a/F$. Then $x/F = (x \vee (x \wedge a))/F = x/F \vee (x/F \wedge a/F) = x/F \vee (y/F \wedge a/F) = (x \vee (y \wedge a))/F$ and, analogously, $y/F = (y \vee (x \wedge a))/F$. Since $(x \vee (y \wedge a)) \wedge a = (x \wedge a) \vee (y \wedge a \wedge a) = (x \wedge a) \vee (y \wedge a) = (y \wedge a) \vee (x \wedge a \wedge a) = (y \vee (x \wedge a)) \wedge a$, it follows that $(x/F, y/F) = ((x \vee (y \wedge a))/F, (y \vee (x \wedge a))/F) \in \sim_G / \sim_F$. Therefore we also have $\sim_{G/F} \subseteq \sim_G / \sim_F$. Hence $\sim_{G/F} = \sim_G / \sim_F$.   □

Lemma 2 *If F and G are filters of L, then the following are equivalent:*

1. $F \cap G = \{1\}$ and $F \vee G = L$;

2. *there exists $e \in \mathcal{B}(L)$ such that $F = [e)$ and $G = [\neg e)$.*

   Dually, if I and J are ideals of L, then the following are equivalent:

- $I \cap J = \{0\}$ and $I \vee J = L$;

- *there exists $e \in \mathcal{B}(L)$ such that $I = (e]$ and $J = (\neg e]$.*

Proof: 2$\Rightarrow$1: If $F = [e)$ and $G = [\neg e)$ for some $e \in \mathcal{B}(L)$, then $F \cap G = [e \vee \neg e) = [1) = \{1\}$ and $F \vee G = [e \wedge \neg e) = [0) = L$.
1$\Rightarrow$2: $F \vee G = L$ iff $0 \in F \vee G$ iff $e \wedge f = 0$ for some $e \in F$ and $f \in G$. Then $e \vee f \in F \cap G = \{1\}$, so $e \vee f = 1$. Hence $e, f \in \mathcal{B}(L)$ and $f = \neg e$. Also, $[e) \subseteq F$ and $[\neg e) = [f) \subseteq G$. Now let $x \in F$; then $x \vee f \in F \cap G = \{1\}$, so $x \vee f = 1$, hence $e = \neg f \leq x$. Therefore $x \in [e)$, hence $F \subseteq [e)$. Therefore $F = [e)$. Similarly, for every $y \in G$, it follows that $e \vee y \in F \vee G = \{1\}$, so $e \vee y = 1$, hence $f = \neg e \leq y$. Therefore $y \in [f)$, hence $G \subseteq [f)$. Therefore $G = [f) = [\neg e)$.   □

Corollary 1   *1. $\mathcal{B}(\mathrm{Filt}(L)) = \{[e) \mid e \in \mathcal{B}(L)\}$;*

2. $\mathcal{B}(\mathrm{Id}(L)) = \{(e] \mid e \in \mathcal{B}(L)\}$.

By applying the functor $\mathcal{B}$ to the previous diagram, we obtain the following Boolean morphisms:

$$\mathcal{B}(\mathrm{Filt}(L))$$

$$\begin{array}{ccc} & \mathcal{B}(u_L) & \mathcal{B}(\Phi_L) \\ \mathcal{B}(L) & & \mathcal{B}(\mathrm{Con}(L)) \\ & \mathcal{B}(v_L) & \mathcal{B}(\Psi_L) \\ & \mathcal{B}(\mathrm{Id}(L)) & \end{array}$$

Proposition 2   *1. $\mathcal{B}(u_L)$ is a Boolean anti–isomorphism.*

2. $\mathcal{B}(v_L)$ is a Boolean isomorphism;

3. $\mathcal{B}(\Phi_L)$ and $\mathcal{B}(\Psi_L)$ are injective Boolean morphisms.

Proof: 1 Since u_L is a bounded lattice anti–morphism, it follows that $\mathcal{B}(u_L)$ is a Boolean anti–morphism. The injectivity of u_L and the definition of the functor $\mathcal{B}$ on morphisms prove that $\mathcal{B}(u_L)$ is injective. Corollary 1, 1, shows that $\mathcal{B}(u_L)$ is surjective.
2 By duality, from 1.
3 Φ_L and Ψ_L are bounded lattice morphisms, so $\mathcal{B}(\Phi_L)$ and $\mathcal{B}(\Psi_L)$ are Boolean morphisms. The fact that Φ_L and Ψ_L are injective, along with the definition of the functor $\mathcal{B}$, show that $\mathcal{B}(\Phi_L)$ and $\mathcal{B}(\Psi_L)$ are injective. $\square$

The previous results are known, but, for the sake of completeness, we have provided proofs for them.

For all $F \in \mathrm{Filt}(L)$, $I \in \mathrm{Id}(L)$ and $\theta \in \mathrm{Con}(L)$, let us define the functions:

- $\delta_F : \mathrm{Filt}(L) \to \mathrm{Filt}(L/F)$, for all $G \in \mathrm{Filt}(L)$, $\delta_F(G) = (F \vee G)/F$;

- $\delta_I : \mathrm{Id}(L) \to \mathrm{Id}(L/I)$, for all $J \in \mathrm{Id}(L)$, $\delta_I(J) = (I \vee J)/I$;

- $\delta_\theta : \mathrm{Con}(L) \to \mathrm{Con}(L/\theta)$, for all $\equiv \in \mathrm{Con}(L)$, $\delta_\theta(\equiv) = (\equiv \vee \theta)/\theta$.

We consider that the fact that, in the above, we can have $F = I = L$, produces no danger of confusion, since everywhere in the following it will be clear to which of the functions above we refer.

Remark 3 *It is straightforward that, for all $F \in \text{Filt}(L)$, $I \in \text{Id}(L)$ and $\theta \in \text{Con}(L)$, δ_F, δ_I and δ_θ are bounded lattice morphisms.*

For instance, in order to prove that δ_F is a bounded lattice morphism between the bounded distributive lattices $(\text{Filt}(L), \cap, \vee, \{1\}, L)$ and $(\text{Filt}(L/F), \cap, \vee, \{1/F\}, L/F)$, we may notice the following: $\delta_F(\{1\}) = (F \vee \{1\})/F = F/F = \{1/F\}$, $\delta_F(L) = (F \vee L)/F = L/F$, and, for all $G, H \in \text{Filt}(L)$, $\delta_F(G \vee H) = (F \vee G \vee H)/F = ((F \vee G) \vee (F \vee H))/F = (F \vee G)/F \vee (F \vee H)/F = \delta_F(G) \vee \delta_F(H)$ and, since the lattice $\text{Filt}(L)$ is distributive, $\delta_F(G \cap H) = (F \vee (G \cap H))/F = ((F \vee G) \cap (F \vee H))/F = (F \vee G)/F \cap (F \vee H)/F = \delta_F(G) \cap \delta_F(H)$.

Definition 1   • *For every $\theta \in \text{Con}(L)$, we say that θ has the* Boolean Lifting Property *(abbreviated* BLP*) iff, for all $a \in L$ such that $a/\theta \in \mathcal{B}(L/\theta)$, there exists $e \in \mathcal{B}(L)$ such that $a/\theta = e/\theta$.*

- *For any $\Omega \subseteq \text{Con}(L)$, we say that L has the Ω–Boolean Lifting Property (abbreviated Ω–BLP) iff every $\theta \in \Omega$ has the BLP.*

- *We say that L has the* Boolean Lifting Property *(abbreviated* BLP*) iff L has the $\text{Con}(L)$–BLP.*

- *For every $F \in \text{Filt}(L)$, we say that F has the* Boolean Lifting Property *(abbreviated* BLP*) iff $\sim_F$ has the BLP.*

- *For every $I \in \text{Id}(L)$, we say that I has the* Boolean Lifting Property *(abbreviated* BLP*) iff $\sim_I$ has the BLP.*

- *We say that L has the* Filt–Boolean Lifting Property *(abbreviated* Filt–BLP*) iff L has the $\Phi_L(\text{Filt}(L))$–BLP.*

- *We say that L has the* Id–Boolean Lifting Property *(abbreviated* Id–BLP*) iff L has the $\Psi_L(\text{Id}(L))$–BLP.*

Remark 4   • *L has the BLP iff every $\theta \in \text{Con}(L)$ has BLP.*

- *L has the Filt–BLP iff every $F \in \text{Filt}(L)$ has BLP.*

- *L has the Id–BLP iff every $I \in \text{Id}(L)$ has BLP.*

Remark 5 *As we have seen in Section 2, for any $\theta \in \text{Con}(L)$, $\mathcal{B}(L)/\theta \subseteq \mathcal{B}(L/\theta)$. Hence, for any $F \in \text{Filt}(L)$, $\mathcal{B}(L)/F \subseteq \mathcal{B}(L/F)$, and, for any $I \in \text{Id}(L)$, $\mathcal{B}(L)/I \subseteq \mathcal{B}(L/I)$.*

For every $\theta \in \mathrm{Con}(L)$, $F \in \mathrm{Filt}(L)$ and $I \in \mathrm{Id}(L)$, let us consider the Boolean morphisms: $\mathcal{B}(p_\theta) : \mathcal{B}(L) \to \mathcal{B}(L/\theta)$, $\mathcal{B}(p_F) : \mathcal{B}(L) \to \mathcal{B}(L/F)$, $\mathcal{B}(p_I) : \mathcal{B}(L) \to \mathcal{B}(L/I)$. The images of these Boolean morphisms are: $\mathcal{B}(p_\theta)(\mathcal{B}(L)) = \mathcal{B}(L)/\theta$, $\mathcal{B}(p_F)(\mathcal{B}(L)) = \mathcal{B}(L)/F$, $\mathcal{B}(p_I)(\mathcal{B}(L)) = \mathcal{B}(L)/I$.

Remark 6 *For all $\theta \in \mathrm{Con}(L)$, $F \in \mathrm{Filt}(L)$ and $I \in \mathrm{Id}(L)$:*

- *θ has BLP iff, for all $a \in L$, $a/\theta \in \mathcal{B}(L/\theta)$ implies $a/\theta \in \mathcal{B}(L)/\theta$, iff $\mathcal{B}(L/\theta) \subseteq \mathcal{B}(L)/\theta$ iff $\mathcal{B}(L/\theta) = \mathcal{B}(L)/\theta$ iff the Boolean morphism $\mathcal{B}(p_\theta)$ is surjective;*

- *F has BLP iff, for all $a \in L$, $a/F \in \mathcal{B}(L/F)$ implies $a/F \in \mathcal{B}(L)/F$, iff $\mathcal{B}(L/F) \subseteq \mathcal{B}(L)/F$ iff $\mathcal{B}(L/F) = \mathcal{B}(L)/F$ iff the Boolean morphism $\mathcal{B}(p_F)$ is surjective;*

- *I has BLP iff, for all $a \in L$, $a/I \in \mathcal{B}(L/I)$ implies $a/I \in \mathcal{B}(L)/I$, iff $\mathcal{B}(L/I) \subseteq \mathcal{B}(L)/I$ iff $\mathcal{B}(L/I) = \mathcal{B}(L)/I$ iff the Boolean morphism $\mathcal{B}(p_I)$ is surjective.*

Remark 7 *Clearly, if $\Omega \subseteq \Sigma \subseteq \mathrm{Con}(L)$ and L has Σ–BLP, then L has Ω–BLP. For instance, if L has BLP, then L has Filt–BLP and Id–BLP.*

Remark 8 *It is clear that the properties Filt–BLP and Id–BLP are dual to each other: L has Filt–BLP iff the dual of L has Id–BLP. Also, clearly, the notion of BLP is dual to itself: L has BLP iff the dual of L has BLP.*

In what follows, we shall be using the previous remarks without referencing them. Actually, throughout this paper, in most cases, remarks will be used without being referenced.

Remark 9 *Any Boolean algebra has BLP. Indeed, if L is a Boolean algebra, then $\mathcal{B}(L) = L$, thus, for any $\theta \in \mathrm{Con}(L)$, $\mathcal{B}(L)/\theta = L/\theta \supseteq \mathcal{B}(L/\theta)$, so θ has BLP.*

Furthermore, by Remark 5, we get that $\mathcal{B}(L)/\theta \subseteq \mathcal{B}(L/\theta)$ and $\mathcal{B}(L)/\theta \supseteq \mathcal{B}(L/\theta)$, so $\mathcal{B}(L/\theta) = \mathcal{B}(L)/\theta = L/\theta$, thus L/θ is a Boolean algebra, as expected, since, in this particular case, θ is a Boolean algebra congruence.

Proposition 3 *For every $F \in \mathrm{Filt}(L)$, the following are equivalent:*

1. *F has BLP;*

2. *the Boolean morphism $\mathcal{B}(\delta_F) : \mathcal{B}(\mathrm{Filt}(L)) \to \mathcal{B}(\mathrm{Filt}(L/F))$ is surjective.*

Dually, for every $I \in \mathrm{Id}(L)$, the following are equivalent:

- *I has BLP;*

- *the Boolean morphism $\mathcal{B}(\delta_I) : \mathcal{B}(\mathrm{Id}(L)) \to \mathcal{B}(\mathrm{Id}(L/I))$ is surjective.*

Proof: The following diagram is commutative:

$$
\begin{array}{ccc}
L & \xrightarrow{\ u_L\ } & \mathrm{Filt}(L) \\
{\scriptstyle p_F}\downarrow & & \downarrow{\scriptstyle \delta_F} \\
L/F & \xrightarrow{\ u_{L/F}\ } & \mathrm{Filt}(L/F)
\end{array}
$$

Indeed, for all $a \in L$, the following hold: $u_{L/F}(p_F(a)) = u_{L/F}(a/F) = [a/F)$ and $\delta_F(u_L(a)) = \delta_F([a)) = ([a) \vee F)/F = [a)/F \vee F/F = [a)/F \vee \{1/F\} = [a)/F = [a/F)$; hence $u_{L/F} \circ p_F = \delta_F \circ u_L$.

By applying the functor $\mathcal{B}$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{B}(L) & \xrightarrow{\ \mathcal{B}(u_L)\ } & \mathcal{B}(\mathrm{Filt}(L)) \\
{\scriptstyle \mathcal{B}(p_F)}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(\delta_F)} \\
\mathcal{B}(L/F) & \xrightarrow{\ \mathcal{B}(u_{L/F})\ } & \mathcal{B}(\mathrm{Filt}(L/F))
\end{array}
$$

that is we get that: $\mathcal{B}(u_{L/F}) \circ \mathcal{B}(p_F) = \mathcal{B}(\delta_F) \circ \mathcal{B}(u_L)$. According to Proposition 2, 1, $\mathcal{B}(u_L)$ and $\mathcal{B}(u_{L/F})$ are bijections. Therefore $\mathcal{B}(p_F)$ is surjective iff $\mathcal{B}(\delta_F)$ is surjective, thus F has BLP iff $\mathcal{B}(\delta_F)$ is surjective. $\square$

Corollary 2 *L has Filt–BLP iff, for each $F \in \mathrm{Filt}(L)$, $\mathcal{B}(\delta_F)$ is surjective. Dually, L has Id–BLP iff, for each $I \in \mathrm{Id}(L)$, $\mathcal{B}(\delta_I)$ is surjective.*

Example 1 *Here is an example of a filter without BLP. Let L be the bounded distributive lattice given by the following Hasse diagram:*

The form of the congruence $\sim_{[c)}$ shows that: $c/[c) = 1/[c) = [c)$, $0/[c) = \{0\}$ and $(a, b) \notin \sim_{[c)}$, that is $a/[c) \neq b/[c)$, hence $a/[c) = \{a\}$ and $b/[c) = \{b\}$. Therefore $L/[c)$ is the rhombus (the direct product between the two–element chain and itself), which is a Boolean algebra, so $\mathcal{B}(L/[c)) = L/[c)$. But, clearly, $\mathcal{B}(L) = \{0, 1\}$, thus $\mathcal{B}(L)/[c) = \{0/[c), 1/[c)\} \subsetneq L/[c) = \mathcal{B}(L/[c))$, therefore the filter $[c)$ does not have BLP. Consequently, L does not have Filt–BLP, hence it does not have BLP.

Example 2 *Here is an example of an ideal without BLP. Let L be the following bounded distributive lattice, which is the dual of the one from Example 1:*

$$
\begin{array}{c}
1 \\
a \bullet \quad \bullet b \\
\bullet c \\
0
\end{array}
$$
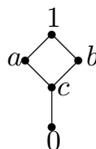
By duality, from Example 1, we get that the ideal $(c]$ does not have BLP in L. Hence L does not have Id–BLP, thus L does not have BLP.

Example 3 *In response to a question posed by the reviewer, let us provide an example of a filter without BLP such that the quotient bounded lattice through that filter is not a Boolean algebra. By dualizing this example, we shall get an ideal with the same property.*

Let L be the lattice with the Hasse diagram below, and let us consider the filter $[a)$ of L:

$$
(L) \qquad\qquad (L/[a))
$$
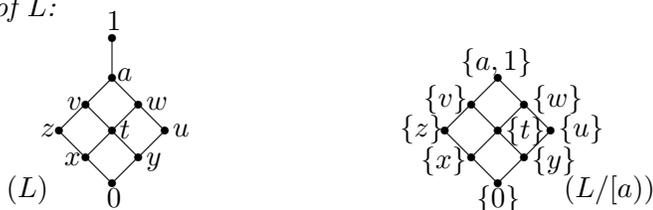
Then $\mathcal{B}(L) = \{0, 1\}$, $[a) = \{a, 1\} = a/[a) = 1/[a)$ and, for all $\alpha, \beta \in L \setminus [a)$ with $\alpha \neq \beta$, we have $\alpha \leq a$ and $\beta \leq a$, thus $\alpha \wedge a = \alpha \neq \beta = \beta \wedge a$, so $\alpha/[a) \neq \beta/[a)$, hence, for all $\alpha \in L \setminus [a)$, $\alpha/[a) = \{\alpha\}$. Thus $L/[a) = \{x/[a) \mid x \in L \setminus \{a\}\}$ is the direct product between the three–element chain and itself, which is not a Boolean algebra, and has $\mathcal{B}(L/[a)] = \{0/[a), z/[a), u/[a), 1/[a)\} \supsetneq \{0/[a), 1/[a)\} = \{0, 1\}/[a) = \mathcal{B}(L)/[a)$, therefore $[a)$ does not have BLP.

Of course, in the dual of L, $(a]$ is an ideal without BLP such that $L/(a]$ is not a Boolean algebra.

Lemma 3 [15, Theorem 2.3, (iii)] *For every $\theta \in \mathrm{Con}(L)$ and all $a, b \in L$,*
$\delta_\theta(Cg_L(a, b)) = (Cg_L(a, b) \vee \theta)/\theta = Cg_{L/\theta}(a/\theta, b/\theta)$.

Proposition 4 *For each $\theta \in \mathrm{Con}(L)$, the Boolean morphism $\mathcal{B}(\delta_\theta) : \mathcal{B}(\mathrm{Con}(L)) \to \mathcal{B}(\mathrm{Con}(L/\theta))$ is surjective.*

Proof: Let $\equiv\, \in \mathcal{B}(\mathrm{Con}(L/\theta))$. Then, according to Proposition 1, 2, there exists $n \in \mathbb{N}^*$ and, for every $i \in \overline{1, n}$, there exist $a_i, b_i \in L$, such that $\equiv\, = \bigvee_{i=1}^{n} Cg_{L/\theta}(a_i/\theta, b_i/\theta)$. Then, by Lemma 3, $\equiv\, = \bigvee_{i=1}^{n} \delta_\theta(Cg_L(a_i, b_i)) = \delta_\theta(\bigvee_{i=1}^{n} Cg_L(a_i, b_i))$. By Proposition 1, 1, for all $i \in \overline{1, n}$, $Cg_L(a_i, b_i) \in \mathcal{B}(\mathrm{Con}(L))$, hence $\bigvee_{i=1}^{n} Cg_L(a_i, b_i) \in \mathcal{B}(\mathrm{Con}(L))$. Therefore we have: $\equiv\, = \mathcal{B}(\delta_\theta)(\bigvee_{i=1}^{n} Cg_L(a_i, b_i)) \in \mathcal{B}(\delta_\theta)(\mathcal{B}(\mathrm{Con}(L)))$. Hence $\mathcal{B}(\delta_\theta)$ is surjective. $\square$

Remark 10 *A characterization similar to the one from Proposition 3 for filters and ideals does not hold for congruences: according to Proposition 4, $\mathcal{B}(\delta_\theta)$ is surjective for every congruence θ of L; but, obviously, not every congruence has BLP. Indeed, if $F = [c)$ is the filter without BLP from Example 1, then the congruence $\sim_F$ does not have BLP. Similarly, if $I = (c]$ is the ideal without BLP from Example 2, then the congruence $\sim_I$ does not have BLP.*

Corollary 3 *If $\mathcal{B}(\Phi_L)$ is surjective, then L has Filt–BLP.*
Dually, if $\mathcal{B}(\Psi_L)$ is surjective, then L has Id–BLP.

Proof: Assume that $\mathcal{B}(\Phi_L)$ is surjective, and let F be an arbitrary filter of L. Then the following diagram is commutative:

$$
\begin{array}{ccc}
\mathrm{Filt}(L) & \xrightarrow{\Phi_L} & \mathrm{Con}(L) \\
\delta_F \downarrow & & \downarrow \delta_{\sim_F} \\
\mathrm{Filt}(L/F) & \xrightarrow{\Phi_{L/F}} & \mathrm{Con}(L/F)
\end{array}
$$

Indeed, for all $G \in \mathrm{Filt}(L)$, the following hold, according to Lemma 1:
$\delta_{\sim_F}(\Phi_L(G)) = \delta_{\sim_F}(\sim_G) = (\sim_F \vee \sim_G)/\sim_F = (\Phi_L(F) \vee \Phi_L(G))/\Phi_L(F) = \Phi_L(F \vee G)/\Phi_L(F) = \sim_{F \vee G}/\sim_F = \sim_{(F \vee G)/F} = \Phi_{L/F}((F \vee G)/F) = \Phi_{L/F}(\delta_F(G))$, thus $\Phi_{L/F} \circ \delta_F = \delta_{\sim_F} \circ \Phi_L$.

$$\mathcal{B}(\mathrm{Filt}(L)) \xrightarrow{\mathcal{B}(\Phi_L)} \mathcal{B}(\mathrm{Con}(L))$$

$$\mathcal{B}(\delta_F) \Big\downarrow \qquad\qquad \Big\downarrow \mathcal{B}(\delta_{\sim_F})$$

$$\mathcal{B}(\mathrm{Filt}(L/F)) \xrightarrow{\mathcal{B}(\Phi_{L/F})} \mathcal{B}(\mathrm{Con}(L/F))$$

By applying the functor $\mathcal{B}$, we get the commutative diagram above, that is we obtain: $\mathcal{B}(\Phi_{L/F}) \circ \mathcal{B}(\delta_F) = \mathcal{B}(\delta_{\sim_F}) \circ \mathcal{B}(\Phi_L)$, which is a surjective Boolean morphism, since $\mathcal{B}(\Phi_L)$ is surjective by the hypothesis and $\mathcal{B}(\delta_{\sim_F})$ is surjective by Proposition 4. But, according to Proposition 2, 3, $\mathcal{B}(\Phi_{L/F})$ is injective. From this, it immediately follows that $\mathcal{B}(\delta_F)$ is surjective, hence F has BLP, by Proposition 3. Therefore L has Filt–BLP. $\qquad\square$

Proposition 5   *1. Let $\theta \in \mathrm{Con}(L)$ such that $\mathcal{B}(L/\theta) = \{0/\theta, 1/\theta\}$. Then θ has BLP.*

2. Let $F \in \mathrm{Filt}(L)$ such that $\mathcal{B}(L/F) = \{0/F, 1/F\}$. Then F has BLP.

3. Let $I \in \mathrm{Id}(L)$ such that $\mathcal{B}(L/I) = \{0/I, 1/I\}$. Then I has BLP.

Proof: 1 $\{0,1\} \subseteq \mathcal{B}(L)$, thus $\mathcal{B}(L/\theta) = \{0/\theta, 1/\theta\} \subseteq \mathcal{B}(L)/\theta$, hence θ has BLP. 2 and 3 follow from 1. $\qquad\square$

Remark 11 *The converse of Proposition 5 does not hold. Indeed, for instance, let L be the cube (with the elements denoted as in the picture below), which is a Boolean algebra, thus it has BLP, which means that all of its congruences have BLP, so all of its filters and all of its ideals have BLP:*
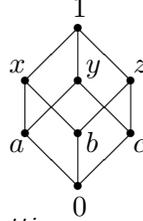
Also, all of its quotient lattices are Boolean algebras, hence, for all $\theta \in \mathrm{Con}(L)$, $\mathcal{B}(L/\theta) = L/\theta$, thus, for all $F \in \mathrm{Filt}(L)$ and all $I \in \mathrm{Id}(L)$, $\mathcal{B}(L/I) = L/I$. Now take, for instance, $[x) = \{x, 1\}$. $L/[x)$ is the rhombus, which has four elements, hence $\mathcal{B}(L/[x)) = L/[x) \supsetneq \{0/[x), 1/[x)\}$; but $[x)$ has BLP in L. Similarly, if we take $(c] = \{0, c\}$, then $L/(c]$ is the rhombus, hence $\mathcal{B}(L/(c]) = L/(c] \supsetneq \{0/(c], 1/(c]\}$; but $(c]$ has BLP. Finally, $\sim_{[x)}$ and $\sim_{(c]}$ have BLP, but $\mathcal{B}(L/\sim_{[x)}) = L/\sim_{[x)} \supsetneq \{0/[x), 1/[x)\} = \{0/\sim_{[x)}, 1/\sim_{[x)}\}$, and the same goes for $\mathcal{B}(L/\sim_{(c]})$.

Corollary 4   *1. Let $\theta \in \mathrm{Con}(L)$. If L/θ is a chain, then θ has BLP.*

    2. Any bounded chain has BLP.

Proof: 1 If the bounded lattice L/θ is a chain, then $\mathcal{B}(L/\theta) = \{0/\theta, 1/\theta\}$, thus θ has BLP by Proposition 5, 1.
2 If L is a bounded chain, then, for every $\theta \in \mathrm{Con}(L)$, the quotient lattice L/θ is a bounded chain as well, hence θ has BLP by 1. Therefore L has BLP. $\square$

Remark 12 *Each of the results Remark 9 and Corollary 4 provides us with a class of counter–examples for the converse of the other one of these results: any bounded chain with at least three elements has BLP, and is not a Boolean algebra, and any Boolean algebra with more than three elements has BLP, and is not a chain.*

Proposition 6   *1. Δ_L and ∇_L have BLP.*

    2. The filters $\{1\}$ and L have BLP.

    3. The ideals $\{0\}$ and L have BLP.

Proof: 1 $L/\Delta_L = \{x/\Delta_L \mid x \in L\} = \{\{x\} \mid x \in L\}$, which is isomorphic to L, since $h : L \to L/\Delta_L$, for all $x \in L$, $h(x) = x/\Delta_L = \{x\}$, is a bounded lattice isomorphism. Hence $\mathcal{B}(L/\Delta_L) = \{\{x\} \mid x \in \mathcal{B}(L)\} = \{x/\Delta_L \mid x \in \mathcal{B}(L)\} = \mathcal{B}(L)/\Delta_L$, thus Δ_L has BLP.
    $L/\nabla_L = \{0/\nabla_L\} = \{1/\nabla_L\} = \{0/\nabla_L, 1/\nabla_L\}$, hence $\mathcal{B}(L/\nabla_L) = L/\nabla_L = \{0/\nabla_L, 1/\nabla_L\}$, therefore ∇_L has BLP by Proposition 5, 1.
2 By 1 and the fact that $\sim_{\{1\}} = \Phi_L(\{1\}) = \Delta_L$ and $\sim_L = \Phi_L(L) = \nabla_L$.
3 By 1 and the fact that $\sim_{\{0\}} = \Psi_L(\{0\}) = \Delta_L$ and $\sim_L = \Psi_L(L) = \nabla_L$, or simply by duality, from 2. $\square$

Proposition 7 *Any prime filter of L has BLP.*
    Dually, any prime ideal of L has BLP.

Proof: Let P be a prime filter of L, and let $x \in L$ such that $x/P \in \mathcal{B}(L/P)$. Then there exists $y \in L$ such that $x/P \vee y/P = 1/P$ and $x/P \wedge y/P = 0/P$. Then $(x \vee y)/P = 1/P = P$, thus $x \vee y \in P$, hence $x \in P$ or $y \in P$ since P is a prime filter. If $x \in P$, then $x/P = 1/P$. If $y \in P$, then $y/P = 1/P$, hence $x/P = x/P \wedge 1/P = x/P \wedge y/P = 0/P$. Since x/P is arbitrary in $\mathcal{B}(L/P)$, it follows that $\mathcal{B}(L/P) = \{0/P, 1/P\}$, hence P has BLP by Proposition 5, 2. $\square$

Corollary 5 *Any maximal filter of L has BLP.*
Dually, any maximal ideal of L has BLP.

Proposition 8   *1. L has* Filt–BLP *iff, for all $F \in \mathrm{Filt}(L)$, L/F has* Filt–BLP*. Moreover, for each $F \in \mathrm{Filt}(L)$, we have: L/F has* Filt–BLP *iff, for all $G \in \mathrm{Filt}(L)$ such that $F \subseteq G$, L/G has* Filt–BLP*.*

   2. L has Id–BLP *iff, for all $I \in \mathrm{Id}(L)$, L/I has* Id–BLP*. Moreover, for each $I \in \mathrm{Id}(L)$, we have: L/I has* Id–BLP *iff, for all $J \in \mathrm{Id}(L)$ such that $I \subseteq J$, L/J has* Id–BLP*.*

   3. L has BLP iff, for all $\theta \in \mathrm{Con}(L)$, L/θ has BLP. Moreover, for each $\theta \in \mathrm{Con}(L)$, we have: L/θ has BLP iff, for all $\phi \in \mathrm{Con}(L)$ such that $\theta \subseteq \phi$, L/ϕ has BLP.

Proof: 1 Assume that L has Filt–BLP and let $F \in \mathrm{Filt}(L)$. Then F has BLP, so $\mathcal{B}(L/F) = \mathcal{B}(L)/F$. Now let us consider an arbitrary filter of L/F, that is let $G \in \mathrm{Filt}(L)$ such that $F \subseteq G$, and let us prove that the filter G/F of L/F has BLP. According to the Second Isomorphism Theorem, the function $h : L/G \to (L/F)/_{(G/F)}$, for all $x \in L$, $h(x/G) = (x/F)/_{(G/F)}$ is a bounded lattice isomorphism. Since L has Filt–BLP, it follows that G has BLP, that is $\mathcal{B}(L/G) = \mathcal{B}(L)/G$. Therefore $\mathcal{B}((L/F))/_{(G/F)} = \{(x/F)/_{(G/F)} \mid x \in L, x/F \in \mathcal{B}(L/F)\} = \{(x/F)/_{(G/F)} \mid x \in \mathcal{B}(L)\} = \{h(x/G) \mid x \in \mathcal{B}(L)\} = h(\mathcal{B}(L)/G) = h(\mathcal{B}(L/G)) = \mathcal{B}((L/F)/_{(G/F)})$, hence G/F has BLP in L/F. Thus L/F has Filt–BLP. For the converse implication, just take $F = \{1\}$.

   The second statement follows from the above, the form of the filters of L/F and the fact that, according to the Second Isomorphism Theorem, for any $G \in \mathrm{Filt}(L)$ such that $F \subseteq G$, the bounded lattice L/G is isomorphic to $(L/F)/_{(G/F)}$.

2 By duality, from 1.

3 The argument is similar to the one from above 1, except filters are replaced by congruences.   □

Remark 13 *The lattice L in Example 2 has* Filt–BLP*, by Proposition 6, 2, Proposition 7 and the immediate fact that all proper filters of L are prime filters. Dually, the lattice L in Example 1 has* Id–BLP*.*

Remark 14 *None of the properties* Filt–BLP *and* Id–BLP *implies the other.*

Indeed, the bounded distributive lattice in Example 2 has Filt–BLP *and it does not have* Id–BLP. *The one in Example 1 has* Id–BLP *and it does not have* Filt–BLP.

Example 4 *Now let us see a bounded distributive lattice which has neither* Filt–BLP, *nor* Id–BLP. *Let L be the following bounded distributive lattice:*

$$(L) \qquad\qquad (L/[a))$$
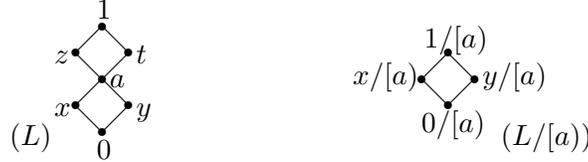
Clearly, $\mathcal{B}(L) = \{0, 1\}$. *Let us consider the filter* $[a) = \{a, z, t, 1\}$. *Then* $a/[a) = z/[a) = t/[a) = 1/[a) = [a)$ *and* $0 \wedge a = 0$, $x \wedge a = x$, $y \wedge a = y$, *thus* $0/[a) = \{0\}$, $x/[a) = \{x\}$, $y/[a) = \{y\}$. *Hence* $L/[a) = \{0/[a), x/[a), y/[a), 1/[a)\}$ *is the rhombus, which is a Boolean algebra, thus* $\mathcal{B}(L/[a)) = L/[a)$, *so* $\mathcal{B}(L)/[a) = \{0/[a), 1/[a)\} \subsetneq L/[a) = \mathcal{B}(L/[a))$, *hence* $[a)$ *does not have BLP, therefore L does not have* Filt–BLP. *At this point, we can notice that, similarly,* $(a]$ *does not have BLP, or we may simply notice that L is dual to itself, hence, since L does not have* Filt–BLP, *it follows that L does not have* Id–BLP *either.*

Until mentioned otherwise, let L_1 and L_2 be bounded distributive lattices and $f : L_1 \to L_2$ be a bounded lattice morphism. For every $\theta \in \mathrm{Con}(L_2)$, we shall denote by $f^{-1}(\theta) = \{(x, y) \mid x, y \in L_1, (f(x), f(y)) \in \theta\}$. For every $\Omega \subseteq \mathrm{Con}(L_2)$, we shall denote by $f^{-1}(\Omega) = \{f^{-1}(\theta) \mid \theta \in \Omega\}$.

Remark 15 *For any* $\theta \in \mathrm{Con}(L_2)$, *it is immediate that* $f^{-1}(\theta) \in \mathrm{Con}(L_1)$.
Thus, for any $\Omega \subseteq \mathrm{Con}(L_2)$, $f^{-1}(\Omega) \subseteq \mathrm{Con}(L_1)$.
Clearly, if f is a bounded lattice isomorphism, then, for every $\theta \in \mathrm{Con}(L_2)$ *and all* $x, y \in L_1$, *we have:* $(x, y) \in f^{-1}(\theta)$ *iff* $(f(x), f(y)) \in \theta$.

Until mentioned otherwise, assume that f is surjective.

Proposition 9   *1. Let* $\theta \in \mathrm{Con}(L_2)$. *If* $f^{-1}(\theta)$ *has BLP (in* L_1), *then* θ *has BLP (in* L_2).

   2. Let $\Omega \subseteq \mathrm{Con}(L_2)$. *If* L_1 *has* $f^{-1}(\Omega)$–BLP, *then* L_2 *has* Ω–BLP.

   3. If L_1 *has* $f^{-1}(\mathrm{Con}(L))$–BLP, *then* L_2 *has BLP.*

   4. If L_1 *has BLP, then* L_2 *has BLP.*

Proof: 1 Assume that $f^{-1}(\theta)$ has BLP, and let $y \in L_2$ such that $y/\theta \in \mathcal{B}(L_2/\theta)$. Then $y/\theta \vee b/\theta = 1/\theta$ and $y/\theta \wedge b/\theta = 0/\theta$ for some $b \in L_2$. Since f is surjective, it follows that there exist $x, a \in L_1$ such that $f(x) = y$ and $f(a) = b$. Hence $f(x)/\theta \vee f(a)/\theta = 1/\theta$ and $f(x)/\theta \wedge f(a)/\theta = 0/\theta$, that is $(f(x) \vee f(a))/\theta = 1/\theta$ and $(f(x) \wedge f(a))/\theta = 0/\theta$, so $f(x \vee a)/\theta = 1/\theta$ and $f(x \wedge a)/\theta = 0/\theta$, which means that $(f(x \vee a), 1) \in \theta$ and $(f(x \wedge a), 0) \in \theta$, that is $(f(x \vee a), f(1)) \in \theta$ and $(f(x \wedge a), f(0)) \in \theta$, thus $(x \vee a, 1) \in f^{-1}(\theta)$ and $(x \wedge a, 0) \in f^{-1}(\theta)$, which means that $(x \vee a)/f^{-1}(\theta) = 1/f^{-1}(\theta)$ and $(x \wedge a)/f^{-1}(\theta) = 0/f^{-1}(\theta)$, so $x/f^{-1}(\theta) \vee a/f^{-1}(\theta) = 1/f^{-1}(\theta)$ and $x/f^{-1}(\theta) \wedge a/f^{-1}(\theta) = 0/f^{-1}(\theta)$, therefore $x/f^{-1}(\theta) \in \mathcal{B}(L_1/f^{-1}(\theta))$. But $f^{-1}(\theta)$ has BLP, hence $x/f^{-1}(\theta) \in \mathcal{B}(L_1)/f^{-1}(\theta)$, that is $x/f^{-1}(\theta) = e/f^{-1}(\theta)$ for some $e \in \mathcal{B}(L_1)$. Then, clearly, $f(e) \in \mathcal{B}(L_2)$, and $(x, e) \in f^{-1}(\theta)$, thus $(f(x), f(e)) \in \theta$, that is $f(x)/\theta = f(e)/\theta \in \mathcal{B}(L_2)/\theta$. Therefore $\mathcal{B}(L_2/\theta) \subseteq \mathcal{B}(L_2)/\theta$, which means that θ has BLP.
2, 3, 4 follow from 1, 2, 3, respectively.   □

Proposition 10 *If f is a bounded lattice isomorphism, then:*

1. *for every $\theta \in \mathrm{Con}(L_2)$, $f^{-1}(\theta)$ has BLP (in L_1) iff θ has BLP (in L_2);*

2. *for any $\Omega \subseteq \mathrm{Con}(L_2)$, L_1 has $f^{-1}(\Omega)$–BLP iff L_2 has Ω–BLP.*

Proof: 1 Let $\theta \in \mathrm{Con}(L_2)$. By Proposition 9, 1, the direct implication holds. Now assume that θ has BLP, and let $x \in L_1$ such that $x/f^{-1}(\theta) \in \mathcal{B}(L_1/f^{-1}(\theta))$. Then $(x \vee a)/f^{-1}(\theta) = x/f^{-1}(\theta) \vee a/f^{-1}(\theta) = 1/f^{-1}(\theta)$ and $(x \wedge a)/f^{-1}(\theta) = x/f^{-1}(\theta) \wedge a/f^{-1}(\theta) = 0/f^{-1}(\theta)$ for some $a \in L_1$. So $(x \vee a, 1) \in f^{-1}(\theta)$ and $(x \wedge a, 0) \in f^{-1}(\theta)$, hence $(f(x \vee a), f(1)) \in \theta$ and $(f(x \wedge a), f(0)) \in \theta$, that is $(f(x) \vee f(a), 1) \in \theta$ and $(f(x) \wedge f(a), 0) \in \theta$, which means that $f(x)/\theta \vee f(a)/\theta = (f(x) \vee f(a))/\theta = 1/\theta$ and $f(x)/\theta \wedge f(a)/\theta = (f(x) \wedge f(a))/\theta = 0/\theta$, hence $f(x)/\theta \in \mathcal{B}(L_2/\theta) = \mathcal{B}(L_2)/\theta$, since θ has BLP. Thus $f(x)/\theta = g/\theta$ for some $g \in \mathcal{B}(L_2) = f(\mathcal{B}(L_1))$, where the last equality is immediate from the fact that f is a bounded lattice isomorphism. Hence $f(x)/\theta = f(e)/\theta$ for some $e \in \mathcal{B}(L_1)$, so $(f(x), f(e)) \in \theta$, thus $(x, e) \in f^{-1}(\theta)$, that is $x/f^{-1}(\theta) = e/f^{-1}(\theta) \in \mathcal{B}(L_1)/f^{-1}(\theta)$. Therefore $\mathcal{B}(L_1/f^{-1}(\theta)) \subseteq \mathcal{B}(L_1)/f^{-1}(\theta)$, so $f^{-1}(\theta)$ has BLP.
2 By 1.   □

Trivially, if the lattices L_1 and L_2 are isomorphic, then: L_1 has BLP, or Filt–BLP, or Id–BLP iff L_2 has BLP, or Filt–BLP, or Id–BLP, respectively.

Proposition 11   • *For all $G \in \text{Filt}(L_2)$, $\sim_{f^{-1}(G)} \subseteq f^{-1}(\sim_G)$; if f is a bounded lattice isomorphism, then, for all $G \in \text{Filt}(L_2)$, $\sim_{f^{-1}(G)} = f^{-1}(\sim_G)$.*

   • *Dually, for all $J \in \text{Id}(L_2)$, $\sim_{f^{-1}(J)} \subseteq f^{-1}(\sim_J)$; if f is a bounded lattice isomorphism, then, for all $J \in \text{Id}(L_2)$, $\sim_{f^{-1}(J)} = f^{-1}(\sim_J)$.*

Proof: Let $x, y \in L_1$ such that $(x, y) \in \sim_{f^{-1}(G)}$, that is $x \wedge a = y \wedge a$ for some $a \in f^{-1}(G)$, hence $f(a) \in G$ and $f(x) \wedge f(a) = f(x \wedge a) = f(y \wedge a) = f(y) \wedge f(a)$, thus $(f(x), f(y)) \in \sim_G$, so $(x, y) \in f^{-1}(\sim_G)$. Therefore $\sim_{f^{-1}(G)} \subseteq f^{-1}(\sim_G)$. Now assume that f is bijective and let $x, y \in L_1$ such that $(x, y) \in f^{-1}(\sim_G)$. Then $(f(x), f(y)) \in \sim_G$, which means that $f(x) \wedge b = f(y) \wedge b$ for some $b \in G$. Since f is surjective, there exists $a \in L_1$ such that $f(a) = b$. But $b \in G$, thus $a \in f^{-1}(G)$. Then we have $f(x) \wedge f(a) = f(y) \wedge f(a)$, so $f(x \wedge a) = f(y \wedge a)$, hence $x \wedge a = y \wedge a$ since f is injective. Therefore $(x, y) \in \sim_{f^{-1}(G)}$. So $f^{-1}(\sim_G) \subseteq \sim_{f^{-1}(G)}$. Therefore $\sim_{f^{-1}(G)} = f^{-1}(\sim_G)$.   □

Corollary 6 *If f is a bounded lattice isomorphism, then:*
for all $G \in \text{Filt}(L_2)$: $f^{-1}(G)$ has BLP (in L_1) iff G has BLP (in L_2);
dually, for all $J \in \text{Id}(L_2)$: $f^{-1}(J)$ has BLP (in L_1) iff J has BLP (in L_2).

Proof: By Proposition 11 and Proposition 10, 1, for any $G \in \text{Filt}(L_2)$, we have: $f^{-1}(G)$ has BLP iff $\sim_{f^{-1}(G)}$ has BLP iff $f^{-1}(\sim_G)$ has BLP iff $\sim_G$ has BLP iff G has BLP.   □

Example 5 *Here is a counter–example for the converse inclusions in Proposition 11, in the general case: let L_1 and L_2 be the bounded distributive lattices given by the next Hasse diagrams, and $f : L_1 \to L_2$ be given by the following table:*

| α	0	a	b	c	d	1
$f(\alpha)$	0	0	x	y	1	1
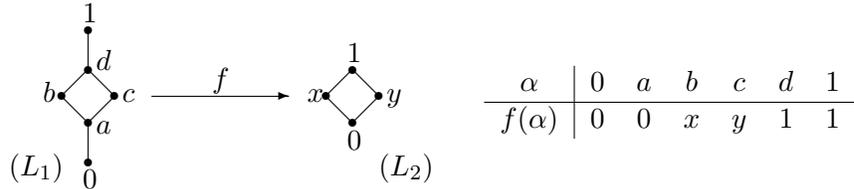
Notice that f is a surjective bounded lattice morphism which is not injective. Let us consider the filter $[y) = \{y, 1\}$ of L_2. $f^{-1}([y)) = \{c, d, 1\} = [c)$. Since $0 \wedge c = 0 \neq a = a \wedge c$, we have $(0, a) \notin \sim_{[c)} = \sim_{f^{-1}([y))}$. But $f(0) = f(a) = 0$, thus $(f(0), f(a)) = (0, 0) \in \sim_{[y)}$, so $(0, a) \in f^{-1}(\sim_{[y)})$. Hence $f^{-1}(\sim_{[y)}) \not\subseteq \sim_{f^{-1}([y))}$. Similarly, if we consider the ideal $(x]$ of L_2, we get $f^{-1}((x]) = (b]$, and we obtain that $f^{-1}(\sim_{(x]}) \not\subseteq \sim_{f^{-1}((x])}$.

Example 6 *Now let us see an example of a bounded distributive lattice which has $\mathrm{Filt-BLP}$ and $\mathrm{Id-BLP}$, but it does not have BLP, and, at the same time, an example of a congruence which does not correspond to a filter or an ideal.*

Let us consider the bounded distributive lattice L_1 from Example 5. This lattice is not a chain, nor a Boolean algebra, nor a direct product of chains (see Corollary 11).

Since L_1 is finite, we have $\mathrm{Filt}(L_1) = \mathrm{PFilt}(L_1)$. By Proposition 6, 2, the filters $[0) = L_1$ and $[1) = \{1\}$ have BLP. $[a) \in \mathrm{Max}_{\mathrm{Filt}}(L_1)$, thus $[a)$ has BLP by Corollary 5. $[b), [c) \in \mathrm{Spec}_{\mathrm{Filt}}(L_1)$, thus $[b)$ and $[c)$ have BLP by Proposition 7.

$$(L_1/[d)) \qquad (L_1/[b)] \qquad (L_1/\theta)$$
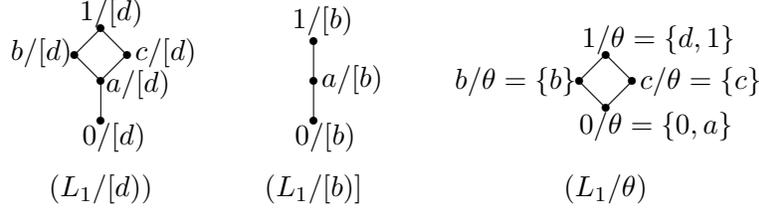
Notice that $\mathcal{B}(L_1) = \{0, 1\}$, and let us consider the filter $[d) = \{d, 1\}$ of L_1. We have: $d/[d) = 1/[d) = [d)$ and each $x \in L_1 \setminus [d)$ satisfies $x \leq d$, thus $x \wedge d = x$, so $0/[d) = \{0\}$, $a/[d) = \{a\}$, $b/[d) = \{b\}$ and $c/[d) = \{c\}$. Hence $L_1/[d)$ is isomorphic to the lattice in Example 2, thus $\mathcal{B}(L_1/[d)) = \{0/[d), 1/[d)\} = \{0, 1\}/[d) = \mathcal{B}(L_1)/[d)$, so $[d)$ has BLP.

Therefore L_1 has $\mathrm{Filt-BLP}$. Since L_1 is dual to itself, it follows that L_1 also has $\mathrm{Id-BLP}$.

Now let us denote by θ the equivalence which corresponds to the following partition of L_1: $\{\{0, a\}, \{b\}, \{c\}, \{d, 1\}\}$. It is easy to see that, for all $x, y, z \in L_1$ such that $(x, y) \in \theta$, it follows that $(x \wedge z, y \wedge z) \in \theta$ and $(x \vee z, y \vee z) \in \theta$; from this it is immediate that $\theta \in \mathrm{Con}(L_1)$. And L_1/θ is the rhombus, which is a Boolean algebra, so $\mathcal{B}(L_1/\theta) = L_1/\theta \supsetneq \{0/\theta, 1/\theta\} = \{0, 1\}/\theta = \mathcal{B}(L_1)/\theta$, thus θ does not have BLP. Hence L_1 does not have BLP.

Notice, additionally, that the congruence θ does not correspond to any filter of L_1, nor to any ideal of L_1, which can be derived from the fact that

L_1 has Filt–BLP and Id–BLP, while θ does not have BLP, but can also be observed directly: if there would exist $F \in \mathrm{Filt}(L)$ such that $\theta = \sim_F$, then we would have $F = 1/\sim_F = 1/\theta = \{d, 1\} = [d)$, so $\theta = \sim_{[d)}$; but $0 \wedge d = 0 \neq a = a \wedge d$, so $(0, a) \notin \sim_{[d)}$, while $0/\theta = a/\theta$, so $(0, a) \in \theta$; we have obtained a contradiction; a similar proof can be given to the fact that the congruence θ does not correspond to any ideal of L_1.

Remark 16 *The following example shows that the converses of the statements from Proposition 9 do not hold, and, moreover, they do not even hold in the particular cases of* Filt–BLP *or* Id–BLP *instead of BLP. This example also shows that the version of statement 4 from Proposition 9 for* Filt–BLP *or* Id–BLP *instead of BLP does not hold either.*

Example 7 *Let L_1, L_2, L_3, L_4, L_5 be the bounded distributive lattices with the following Hasse diagrams:*

(L_1) (L_2) (L_3) (L_4) (L_5)
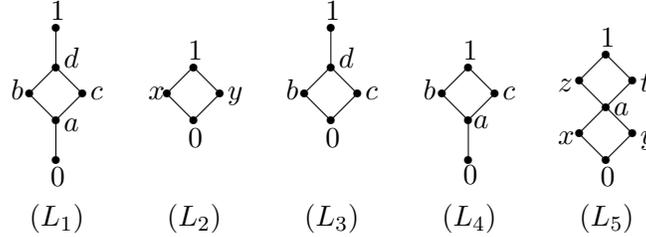
So L_1 is the lattice L_1 in Example 5, L_2 is the rhombus and L_3, L_4 and L_5 are the lattices in Examples 1, 2 and 4, respectively.

Notice that the functions $f : L_1 \to L_2$, $g : L_1 \to L_3$, $h : L_1 \to L_4$, $k : L_3 \to L_2$, $l : L_4 \to L_2$ and $m : L_5 \to L_2$, defined by the following tables, are surjective bounded lattice morphisms:

α	0	a	b	c	d	1
$f(\alpha)$	0	0	x	y	1	1
$g(\alpha)$	0	0	b	c	d	1
$h(\alpha)$	0	a	b	c	1	1
$k(\alpha)$	0		x	y	1	1
$l(\alpha)$	0	0	x	y		1

α	0	x	y	a	z	t	1
$m(\alpha)$	0	x	y	1	1	1	1

And now let us notice that, although there exist surjective bounded lattice morphisms between these bounded distributive lattices, we have:

(a) as shown in Example 6, L_1 does not have BLP, while L_2 is the rhombus, which is a Boolean algebra, hence it has BLP;
(b) L_3 does not have Filt–BLP, while L_2 has BLP and thus Filt–BLP;
(c) L_4 does not have Id–BLP, while L_2 has BLP and thus Id–BLP;
(d) as shown in Example 6, L_1 has Filt–BLP, while L_3 does not have Filt–BLP;
(e) as shown in Example 6, L_1 has Id–BLP, while L_4 does not have Id–BLP.
(f) L_5 has neither Filt–BLP, nor Id–BLP, while L_2 is a Boolean algebra, thus it has BLP; this observation strengthens (a) above.

4   Characterization Theorems for the Boolean Lifting Properties

In this section, we prove that the Boolean Lifting Property for filters is equivalent to B–conormality, and, when the Boolean center is the two–element chain, also to Filt–locality. Dually, the Boolean Lifting Property for ideals is equivalent to B–normality, and, when the Boolean center is the two–element chain, also to Id–locality. We also prove that the Boolean Lifting Properties are preserved by arbitrary direct products, and we provide a method for obtaining examples of bounded distributive lattices which have these Boolean Lifting Properties and examples without these properties.

Throughout this section, unless mentioned otherwise, L will be an arbitrary bounded distributive lattice.

We recall that L is said to be:

- *B–normal* iff, for all $x, y \in L$ such that $x \vee y = 1$, there exist $e, f \in \mathcal{B}(L)$ such that $e \wedge f = 0$ and $x \vee e = y \vee f = 1$;

- *B–conormal* iff, for all $x, y \in L$ such that $x \wedge y = 0$, there exist $e, f \in \mathcal{B}(L)$ such that $e \vee f = 1$ and $x \wedge e = y \wedge f = 0$.

Clearly, the notions of B–normality and B–conormality are dual to each other, that is L is B–normal iff its dual is B–conormal.

Lemma 4 L is B–normal iff, for all $x, y \in L$ such that $x \vee y = 1$, there exists $e \in \mathcal{B}(L)$ such that $x \vee e = y \vee \neg e = 1$;

Dually, L is B–conormal iff, for all $x, y \in L$ such that $x \wedge y = 0$, there exists $e \in \mathcal{B}(L)$ such that $x \wedge e = y \wedge \neg e = 0$.

Proof: Assume that L is B–conormal, and let $x, y \in L$ such that $x \wedge y = 0$. Then there exist $e, f \in \mathcal{B}(L)$ such that $e \vee f = 1$ and $x \wedge e = y \wedge f = 0$. $e \vee f = 1$ means that $f \geq \neg e$. Hence $y \wedge \neg e \leq y \wedge f = 0$, thus $y \wedge \neg e = 0$. The converse implication is trivial. $\hfill\square$

Proposition 12 *The following statements are equivalent:*

1. *L has* Filt*–BLP;*

2. *L is B–conormal;*

3. $\mathrm{PFilt}(L)$ *is B–normal;*

4. $\mathrm{PId}(L)$ *is B–conormal;*

5. $\mathrm{Filt}(L)$ *is B–normal;*

6. $\mathrm{Id}(L)$ *is B–conormal;*

7. *for all $n \in \mathbb{N}^*$ and all $x_1, \ldots, x_n \in L$ such that $\bigwedge_{i=1}^{n} x_i = 0$, there exist*

   $e_1, \ldots, e_n \in \mathcal{B}(L)$ such that $\bigwedge_{i=1}^{n} e_i = 0$, $e_i \vee e_j = 1$ for all $i, j \in \overline{1, n}$ with $i \neq j$, and $x_i \leq e_i$ for all $i \in \overline{1, n}$.

   Dually: L has Id*–BLP iff L is B–normal iff $\mathrm{PId}(L)$ is B–normal iff $\mathrm{PFilt}(L)$ is B–conormal iff $\mathrm{Id}(L)$ is B–normal iff $\mathrm{Filt}(L)$ is B–conormal iff: for all $n \in \mathbb{N}^*$ and all $x_1, \ldots, x_n \in L$ such that $\bigvee_{i=1}^{n} x_i = 1$, there exist $e_1, \ldots, e_n \in \mathcal{B}(L)$ such that $\bigvee_{i=1}^{n} e_i = 1$, $e_i \wedge e_j = 0$ for all $i, j \in \overline{1, n}$ with $i \neq j$, and $x_i \geq e_i$ for all $i \in \overline{1, n}$.*

Proof: $1 \Rightarrow 2$: Let $x, y \in L$ such that $x \wedge y = 0$, and let us denote by $F = [x \vee y)$. Then $x/F \wedge y/F = (x \wedge y)/F = 0/F$ and, since $x \vee y \in F$, $x/F \vee y/F = (x \vee y)/F = 1/F$. Thus $x/F, y/F \in \mathcal{B}(L/F)$ and $y/F = \neg x/F$ in the Boolean algebra $\mathcal{B}(L/F)$. But L has Filt–BLP, so F has BLP, thus $\mathcal{B}(L/F) = \mathcal{B}(L)/F$. Hence there exists $e \in \mathcal{B}(L)$ such that $x/F = e/F$, that is $(x, e) \in \sim_F = \sim_{[x \vee y)}$, which means that $x = x \wedge (x \vee y) = e \wedge (x \vee y)$.

$y/F = \neg(x/F) = \neg(e/F) = \neg e/F$, thus $(y, \neg e) \in \sim_F = \sim_{[x \vee y)}$, so $y = y \wedge (x \vee y) = \neg e \wedge (x \vee y)$. We thus have the following: $x \wedge \neg e = e \wedge (x \vee y) \wedge \neg e = e \wedge \neg e \wedge (x \vee y) = 0 \wedge (x \vee y) = 0$ and $y \wedge e = \neg e \wedge (x \vee y) \wedge e = \neg e \wedge e \wedge (x \vee y) = 0 \wedge (x \vee y) = 0$. By Lemma 4, it follows that L is B–conormal.

$2 \Rightarrow 1$: Let $F \in \mathrm{Filt}(L)$ and $x \in L$ such that $x/F \in \mathcal{B}(L/F)$, hence $x/F \vee y/F = 1/F$ and $x/F \wedge y/F = 0/F$ for some $y \in L$. So $(x \wedge y)/F = 0/F$, which means that $(x \wedge y, 0) \in \sim_F$, that is there exists $t \in F$ such that $x \wedge y \wedge t = 0 \wedge t = 0$. Hence $(x \wedge t) \wedge (y \wedge t) = 0$, so, since L is B–conormal, by Lemma 4, it follows that there exists $e \in \mathcal{B}(L)$ such that $x \wedge t \wedge e = y \wedge t \wedge \neg e = 0$. $t \in F$, so $t/F = 1/F$. Therefore $x/F \wedge e/F = x/F \wedge 1/F \wedge e/F = x/F \wedge t/F \wedge e/F = (x \wedge t \wedge e)/F = 0/F$ and $y/F \wedge \neg e/F = y/F \wedge 1/F \wedge \neg e/F = y/F \wedge t/F \wedge \neg e/F = (y \wedge t \wedge \neg e)/F = 0/F$. Hence $x/F \vee e/F = x/F \vee e/F \vee 0/F = x/F \vee e/F \vee (y/F \wedge \neg e/F) = (x/F \vee e/F \vee y/F) \wedge (x/F \vee e/F \vee \neg e/F) = (x/F \vee e/F \vee y/F) \wedge (x/F \vee e/F \vee \neg e/F) = (x/F \vee y/F \vee e/F) \wedge (x/F \vee 1/F) = (1/F \vee e/F) \wedge 1/F = 1/F \wedge 1/F = 1/F$. We have obtained that $x/F \wedge e/F = 0/F$ and $x/F \vee e/F = 1/F$, which means that $x/F = \neg(e/F) = \neg e/F \in \mathcal{B}(L)/F$. Therefore $\mathcal{B}(L/F) \subseteq \mathcal{B}(L)/F$, so F has BLP. Hence L has Filt–BLP.

$2 \Rightarrow 7$: We shall apply induction on $n \in \mathbb{N}^*$. For $n = 1$, the statement is trivial: for $x = 0 \in L$, if we take $e = 0 \in \mathcal{B}(L)$, then $x \leq e$. Now assume that the statement is valid for an $n \in \mathbb{N}^*$ and let us consider $x_1, \ldots, x_n, x_{n+1} \in L$ such that $\bigwedge_{i=1}^{n+1} x_i = 0$. Let us denote by $x = \bigwedge_{i=1}^{n} x_i$. Then $x \wedge x_{n+1} = 0$, hence, since L is B–conormal, by Lemma 4, it follows that there exists an $e \in \mathcal{B}(L)$ such that $x \wedge e = x_{n+1} \wedge \neg e = 0$. Thus $0 = x \wedge e = \bigwedge_{i=1}^{n} x_i \wedge e = \bigwedge_{i=1}^{n}(x_i \wedge e)$. By the induction hypothesis, it follows that there exist $f_1, \ldots, f_n \in \mathcal{B}(L)$ such that $\bigwedge_{i=1}^{n} f_i = 0$, $f_i \vee f_j = 1$ for all $i, j \in \overline{1, n}$ with $i \neq j$, and $x_i \wedge e \leq f_i$ for all $i \in \overline{1, n}$. The fact that $x_{n+1} \wedge \neg e = 0$ is equivalent to $x_{n+1} \leq \neg\neg e = e$. For all $i \in \overline{1, n}$, $x_i \wedge e \wedge \neg f_i \leq f_i \wedge \neg f_i = 0$, so $x_i \wedge e \wedge \neg f_i = 0$, which is equivalent to $x_i \leq \neg(e \wedge \neg f_i) = \neg e \vee \neg\neg f_i = \neg e \vee f_i$. For all $i \in \overline{1, n}$, let $e_i = \neg e \vee f_i \in \mathcal{B}(L)$, and let $e_{n+1} = e$. Then, for all $i \in \overline{1, n+1}$, $x_i \leq e_i$. $\bigwedge_{i=1}^{n+1} e_i = \bigwedge_{i=1}^{n} e_i \wedge e_{n+1} = \bigwedge_{i=1}^{n}(\neg e \vee f_i) \wedge e = (\neg e \vee (\bigwedge_{i=1}^{n} f_i)) \wedge e = (\neg e \vee 0) \wedge e = \neg e \wedge e = 0$. For all $i \in \overline{1, n}$, $e_i \vee e_{n+1} = \neg e \vee f_i \vee e = \neg e \vee e \vee f_i = 1 \vee f_i = 1$. For

all $i, j \in \overline{1, n}$ with $i \neq j$, $e_i \vee e_j = \neg e \vee f_i \vee \neg e \vee f_j = \neg e \vee f_i \vee f_j = \neg e \vee 1 = 1$.
    Hence the statement is valid for all $n \in \mathbb{N}^*$.

$7 \Rightarrow 2$: Let $x, y \in L$ such that $x \wedge y = 0$. Then, by the statement applied for $n = 2$, we get that there exist $e, f \in \mathcal{B}(L)$ such that $e \wedge f = 0$, $e \vee f = 1$, $x \leq e$ and $y \leq f$. Then $f = \neg e$, so we have: $x \leq e$ and $y \leq \neg e$, hence $x \wedge \neg e = y \wedge e = 0$, thus L is B–conormal by Lemma 4.

$2 \Leftrightarrow 3$: By the fact that the bounded distributive lattices L and $\mathrm{PFilt}(L)$ are anti–isomorphic.

$2 \Leftrightarrow 4$: By the fact that the bounded distributive lattices L and $\mathrm{PId}(L)$ are isomorphic.

$2 \Rightarrow 5$: Let $F, G \in \mathrm{Filt}(L)$ such that $F \vee G = L$, thus $x \wedge y = 0$ for some $x \in F$ and $y \in G$. Since L is B–conormal, we get that there exist $e, f \in \mathcal{B}(L)$ such that $e \vee f = 1$ and $x \wedge e = y \wedge f = 0$. Then $[e), [f) \in \mathcal{B}(\mathrm{Filt}(L))$ by Corollary 1, 1, $[e) \cap [f) = [e \vee f) = [1) = \{1\}$ and $[x) \vee [e) = [x \wedge e) = [0) = L = [0) = [y \wedge f) = [y) \vee [f)$, hence $F \vee [e) = G \vee [f) = L$ since $[x) \subseteq F$ and $[y) \subseteq G$. Therefore $\mathrm{Filt}(L)$ is B–normal.

$5 \Rightarrow 2$: Let $x, y \in L$ such that $x \wedge y = 0$, so $[x) \vee [y) = [x \wedge y) = [0) = L$. Since $\mathrm{Filt}(L)$ is B–normal, it follows that there exist $F, G \in \mathcal{B}(\mathrm{Filt}(L))$ such that $F \cap G = \{1\}$ and $[x) \vee F = [y) \vee G = L$. By Corollary 1, 1, it follows that there exist $e, f \in \mathcal{B}(L)$ such that $F = [e)$ and $G = [f)$. Hence $[e \vee f) = [e) \cap [f) = \{1\}$ and $[x \wedge e) = [x) \vee [e) = L = [y) \vee [f) = [y \wedge f)$, so $e \vee f = 1$ and $x \wedge e = y \wedge f = 0$. Therefore L is B–conormal.

$4 \Leftrightarrow 6$: By duality, from the equivalence between 3 and 5.   □

Corollary 7 *The following statements are equivalent:*

- *L has Filt–BLP and Id–BLP;*

- *L is both B–normal and B–conormal.*

Corollary 8 *If L has BLP, then L is both B–normal and B–conormal.*

Corollary 9 *Let us consider the bounded distributive lattices L, $\mathrm{PFilt}(L)$, $\mathrm{PId}(L)$, $\mathrm{Filt}(L)$ and $\mathrm{Id}(L)$.*

- *If either of these lattices is both B–normal and B–conormal, then each of these lattices is both B–normal and B–conormal.*

- *If either of these lattices has BLP, then each of these lattices is both B–normal and B–conormal.*

Corollary 10 *L has* Filt–*BLP iff* $\mathrm{PFilt}(L)$ *has* Id–*BLP iff* $\mathrm{PId}(L)$ *has* Filt–*BLP iff* $\mathrm{Filt}(L)$ *has* Id–*BLP iff* $\mathrm{Id}(L)$ *has* Filt–*BLP.*

Dually, L has Id–*BLP iff* $\mathrm{PId}(L)$ *has* Id–*BLP iff* $\mathrm{PFilt}(L)$ *has* Filt–*BLP iff* $\mathrm{Id}(L)$ *has* Id–*BLP iff* $\mathrm{Filt}(L)$ *has* Filt–*BLP.*

Remark 17 *In view of Corollary 10, it becomes natural to investigate the relation between the presence of the different Boolean Lifting Properties in L and their presence in $\mathrm{Con}(L)$. In this case, however, we find that correlations similar to those in Corollary 10 do not exist.*

For instance, according to Proposition 1, 3, if L is finite, then $\mathrm{Con}(L)$ is a Boolean algebra, thus $\mathrm{Con}(L)$ has BLP. But, in Examples 1, 2, 3 and 6, we have finite distributive lattices without BLP. Moreover, in Example 4, we have a finite distributive lattice which has neither Filt–*BLP, nor* Id–*BLP.*

Proposition 13 *Let $(L_t)_{t\in T}$ be a non–empty family of bounded distributive lattices and $L = \prod_{t\in T} L_t$. Then:*

- *L has* Filt–*BLP iff, for all $t \in T$, L_t has* Filt–*BLP;*

- *dually, L has* Id–*BLP iff, for all $t \in T$, L_t has* Id–*BLP.*

Proof: According to Proposition 12, it suffices to show that: L is B–conormal iff, for all $t \in T$, L_t is B–conormal.

Assume that, for all $t \in T$, L_t is B–conormal, and let $x, y \in L$ such that $x \wedge y = 0$. Then $x = (x_t)_{t\in T}$, $y = (y_t)_{t\in T}$, with $x_t, y_t \in L_t$ for all $t \in T$, and $(0)_{t\in T} = 0 = x \wedge y = (x_t)_{t\in T} \wedge (y_t)_{t\in T} = (x_t \wedge y_t)_{t\in T}$, thus $x_t \wedge y_t = 0$ for all $t \in T$. Since each L_t is B–conormal, it follows that, for all $t \in T$, there exist $e_t, f_t \in \mathcal{B}(L_t)$ such that $e_t \vee f_t = 1$ and $x_t \wedge e_t = y_t \wedge f_t = 0$. Let $e = (e_t)_{t\in T} \in \mathcal{B}(L)$ and $f = (f_t)_{t\in T} \in \mathcal{B}(L)$. Then $e \vee f = (e_t)_{t\in T} \vee (f_t)_{t\in T} = (e_t \vee f_t)_{t\in T} = (1)_{t\in T} = 1$, $x \wedge e = (x_t)_{t\in T} \wedge (e_t)_{t\in T} = (x_t \wedge e_t)_{t\in T} = (0)_{t\in T} = 0$ and $y \wedge f = (y_t)_{t\in T} \wedge (f_t)_{t\in T} = (y_t \wedge f_t)_{t\in T} = (0)_{t\in T} = 0$. Therefore L is B–conormal.

Now assume that L is B–conormal and let $k \in I$, arbitrary.   Let $x_k, y_k \in L_k$ such that $x_k \wedge y_k = 0$. For every $t \in T \setminus \{k\}$, let $x_t = y_t = 0 \in L_t$, and let $x = (x_t)_{t\in T} \in L$ and $y = (y_t)_{t\in T} \in L$.   Then, for all $t \in T$, $x_t \wedge y_t = 0$, so $x \wedge y = (x_t)_{t\in T} \wedge (y_t)_{t\in T} = (x_t \wedge y_t)_{t\in T} = (0)_{t\in T} = 0$. But L is B–conormal, thus there exist $e, f \in \mathcal{B}(L)$ such that $e \vee f = 1$ and $x \wedge e = y \wedge f = 0$. Then $e = (e_t)_{t\in T}$ and $f = (f_t)_{t\in T}$, with $e_t, f_t \in \mathcal{B}(L_t)$

for all $t \in T$. Also, $(e_t \vee f_t)_{t \in T} = (e_t)_{t \in T} \vee (f_t)_{t \in T} = e \vee f = 1 = (1)_{t \in T}$, $(x_t \wedge e_t)_{t \in T} = (x_t)_{t \in T} \wedge (e_t)_{t \in T} = x \wedge e = 0 = (0)_{t \in T}$ and $(y_t \wedge f_t)_{t \in T} = (y_t)_{t \in T} \wedge (f_t)_{t \in T} = y \wedge f = 0 = (0)_{t \in T}$. Thus $e_k, f_k \in \mathcal{B}(L_k)$, $e_k \vee f_k = 1$ and $x_k \wedge e_k = y_k \wedge f_k = 0$. Therefore L_k is B–conormal. $\square$

Proposition 13 does hold for congruences, and it also holds for individual congruences, filters and ideals, but, in these cases, it needs a different proof.

Proposition 14 *Let $(L_t)_{t \in T}$ be a non–empty family of bounded distributive lattices, $L = \prod\limits_{t \in T} L_t$, for all $t \in T$, $\theta_t \in \mathrm{Con}(L_t)$, $F_t \in \mathrm{Filt}(L_t)$ and $I_t \in \mathrm{Id}(L_t)$, $\theta = \prod\limits_{t \in T} \theta_t$, $F = \prod\limits_{t \in T} F_t$ and $I = \prod\limits_{t \in T} I_t$. Then the following hold:*

1. *θ has BLP (in L) iff, for all $t \in T$, θ_t has BLP (in L_t);*

2. *F has BLP (in L) iff, for all $t \in T$, F_t has BLP (in L_t);*

3. *I has BLP (in L) iff, for all $t \in T$, I_t has BLP (in L_t).*

Proof: 1 First let us prove the converse implication, so assume that, for all $t \in T$, the θ_t has BLP, that is $\mathcal{B}(L/\theta_t) = \mathcal{B}(L)/\theta_t$. Let $a \in L$ such that $a/\theta \in \mathcal{B}(L/\theta)$. Then $(a \vee b)/\theta = a/\theta \vee b/\theta = 1/\theta$ and $(a \wedge b)/\theta = a/\theta \wedge b/\theta = 0/\theta$ for some $b \in L$. So $(a \vee b, 1) \in \theta$ and $(a \wedge b, 0) \in \theta$. Let $a = (a_t)_{t \in T}$ and $b = (b_t)_{t \in T}$, with $a_t, b_t \in L_t$ for all $t \in T$. Then $a \vee b = (a_t \vee b_t)_{t \in T}$, $a \wedge b = (a_t \wedge b_t)_{t \in T}$ and, of course, $1 = (1)_{t \in T}$ and $0 = (0)_{t \in T}$. Thus, for all $t \in T$, $(a_t \vee b_t, 1) \in \theta_t$ and $(a_t \wedge b_t, 0) \in \theta_t$, which means that $a_t/\theta_t \vee b_t/\theta_t = (a_t \vee b_t)/\theta_t = 1/\theta_t$ and $a_t/\theta_t \wedge b_t/\theta_t = (a_t \wedge b_t)/\theta_t = 0/\theta_t$, hence $a_t/\theta_t \in \mathcal{B}(L_t/\theta_t) = \mathcal{B}(L_t)/\theta_t$ since θ_t has BLP, so there exists $e_t \in \mathcal{B}(L_t)$ such that $a_t/\theta_t = e_t/\theta_t$. Let $e = (e_t)_{t \in T} \in \prod\limits_{t \in T} \mathcal{B}(L_t) = \mathcal{B}(\prod\limits_{t \in T} L_t) = \mathcal{B}(L)$. Then $(a, e) = ((a_t)_{t \in T}, (e_t)_{t \in T}) \in \prod\limits_{t \in T} \theta_t = \theta$, so $a/\theta = e/\theta \in \mathcal{B}(L)/\theta$. Therefore $\mathcal{B}(L/\theta) \subseteq \mathcal{B}(L)/\theta$, thus θ has BLP.

And now let us prove the direct implication, so assume that θ has BLP, and let $k \in T$. Let $a_k \in L_k$ such that $a_k/\theta_k \in \mathcal{B}(L_k/\theta_k)$, thus $(a_k \vee b_k)/\theta_k = a_k/\theta_k \vee b_k/\theta_k = 1/\theta_k$ and $(a_k \wedge b_k)/\theta_k = a_k/\theta_k \wedge b_k/\theta_k = 0/\theta_k$ for some $b_k \in L_k$. Hence $(a_k \vee b_k, 1) \in \theta_k$ and $(a_k \wedge b_k, 0) \in \theta_k$. Let $a = (x_t)_{t \in T} \in L$ and $b = (y_t)_{t \in T} \in L$, with $x_k = a_k \in L_k$, $y_k = b_k \in L_k$ and, for all $t \in T \setminus \{k\}$, $x_t = 0 \in L_t$ and $y_t = 1 \in L_t$. Then $a \vee b = (x_t \vee y_t)_{t \in T}$, $a \wedge b = (x_t \wedge y_t)_{t \in T}$

and we have: $(x_k \vee y_k, 1) = (a_k \vee b_k, 1) \in \theta_k$, $(x_k \wedge y_k, 0) = (a_k \wedge b_k, 0) \in \theta_k$ and, for all $t \in T \setminus \{k\}$, $(x_t \vee y_t, 1) = (1, 1) \in \theta_t$ and $(x_t \wedge y_t, 0) = (0, 0) \in \theta_t$, therefore $(a \vee b, 1) \in \prod_{t \in T} \theta_t = \theta$ and $(a \wedge b, 0) \in \prod_{t \in T} \theta_t = \theta$, which means that $a/\theta \vee b/\theta = (a \vee b)/\theta = 1/\theta$ and $a/\theta \wedge b/\theta = (a \wedge b)/\theta = 0/\theta$, hence $a/\theta \in \mathcal{B}(L/\theta) = \mathcal{B}(L)/\theta$ since θ has BLP. So $a/\theta = e/\theta$ for some $e \in \mathcal{B}(L) = \prod_{t \in T} \mathcal{B}(L_t)$, which means that $e = (e_t)_{t \in T}$, with $e_t \in \mathcal{B}(L_t)$ for all $t \in T$, and we obtain: $((x_t)_{t \in T}, (e_t)_{t \in T}) = (a, e) \in \theta = \prod_{t \in T} \theta_t$, that is $(x_t, e_t) \in \theta_t$ for all $t \in T$, hence $(a_k, e_k) = (x_k, e_k) \in \theta_k$, thus $a_k/\theta_k = e_k/\theta_k \in \mathcal{B}(L_k)/\theta_k$. Therefore $\mathcal{B}(L_k/\theta_k) \subseteq \mathcal{B}(L_k)/\theta_k$, so θ_k has BLP. Hence we have obtained that, for all $t \in T$, θ_t has BLP.

2 By 1 and the fact that $\sim_F = \prod_{t \in T} \sim_{F_t}$.

3 By 1 and the fact that $\sim_I = \prod_{t \in T} \sim_{I_t}$. $\square$

Proposition 15 *Let $(L_t)_{t \in T}$ be a non–empty family of bounded distributive lattices and $L = \prod_{t \in T} L_t$. Then: L has BLP iff, for all $t \in T$, L_t has BLP.*

Proof: By Proposition 14, 1, and the fact that $\mathrm{Con}(L) = \{\prod_{t \in T} \theta_t \mid (\forall t \in T)\, (\theta_t \in \mathrm{Con}(L_t))\}$. $\square$

Notice, additionally, that Proposition 3, 1 and 2, follow from Proposition 14, 2 and 3, respectively.

Corollary 11 *Any direct product of bounded chains has BLP.*

Proof: By Corollary 4, 2, and Proposition 15. $\square$

Remark 18 *Corollary 11 provides us with a class of examples of bounded distributive lattices with BLP which are neither chains, nor Boolean algebras: any direct product of at least two non–trivial bounded chains such that at least one of these bounded chains has at least three elements has BLP, but it is neither a chain, nor a Boolean algebra. For instance, the direct product between the three–element chain and itself is a bounded distributive lattice with BLP which is neither a chain, nor a Boolean algebra.*

Corollary 12 *If $e \in \mathcal{B}(L)$, then the following hold:*

1. *L has Filt–BLP iff the lattices $[e)$ and $[\neg e)$ have Filt–BLP iff the lattices $(e]$ and $(\neg e]$ have Filt–BLP;*

2. *L has Id–BLP iff the lattices $[e)$ and $[\neg e)$ have Id–BLP iff the lattices $(e]$ and $(\neg e]$ have Id–BLP;*

3. *L has BLP iff the lattices $[e)$ and $[\neg e)$ have BLP iff the lattices $(e]$ and $(\neg e]$ have BLP.*

Proof: By the fact that L is isomorphic to $[e) \times [\neg e)$ and to $(e] \times (\neg e]$ and by: Proposition 13 in the case of 1 and 2, and Proposition 15 in the case of 3. □

Definition 2 [5] *L is said to be Filt–local iff it has a unique maximal filter. L is said to be Id–local iff it has a unique maximal ideal.*

Clearly, the trivial bounded lattice is neither Filt–local, nor Id–local. Clearly, L is Filt–local iff its dual is Id–local, that is these two notions are dual to each other.

Example 8 *The lattice in Example 2 is Filt–local, with the unique maximal filter $[c)$. The one in Example 1 is Id–local, with the unique maximal ideal $(c]$.*

Lemma 5 [5] *If L is non–trivial, then:*

- *L is Filt–local iff, for all $x, y \in L$, $x \wedge y = 0$ implies $x = 0$ or $y = 0$;*

- *L is Id–local iff, for all $x, y \in L$, $x \vee y = 1$ implies $x = 1$ or $y = 1$.*

Corollary 13 *If L is Filt–local or Id–local, then $\mathcal{B}(L) = \{0, 1\}$.*

Corollary 14 *L is Filt–local iff $L \setminus \{0\} \in \mathrm{Filt}(L)$ iff $L \setminus \{0\} \in \mathrm{Max}_{\mathrm{Filt}}(L)$ iff $\mathrm{Max}_{\mathrm{Filt}}(L) = \{L \setminus \{0\}\}$.*
Dually, L is Id–local iff $L \setminus \{1\} \in \mathrm{Id}(L)$ iff $L \setminus \{1\} \in \mathrm{Max}_{\mathrm{Id}}(L)$ iff $\mathrm{Max}_{\mathrm{Id}}(L) = \{L \setminus \{1\}\}$.

Remark 19 *Any non–trivial bounded chain is both* Filt*–local and* Id*–local. Indeed, if L is a non–trivial bounded chain, then $\wedge = \min$ in L, from which it follows that L is* Filt*–local by Lemma 5, and $\vee = \max$ in L, from which it follows that L is* Id*–local, again by Lemma 5.*

It is immediate, for instance from Lemma 5, that a direct product of at least two non–trivial bounded chains is neither Filt*–local, nor* Id*–local. It is straightforward, by Lemma 5, that any Boolean algebra with more than three elements is neither* Filt*–local, nor* Id*–local. In fact, the two–element chain is* Filt*–local and* Id*–local, and it is the only Boolean algebra which has either of these properties.*

Corollary 15 *If L is* Filt*–local, then L has* Filt*–BLP.*
    Dually, if L is Id*–local, then L has* Id*–BLP.*

Proof: Assume that L is Filt–local, and let $x, y \in L$ such that $x \wedge y = 0$. Then $x = 0$ or $y = 0$. Assume, for instance, that $x = 0$. Then, by taking $e = 1$ and $f = 0$, we get: $e, f \in \mathcal{B}(L)$, $e \vee f = 1$ and $x \wedge e = y \wedge f = 0$. Thus L is B–conormal, hence L has Filt–BLP by Proposition 12.   □

Remark 20 *The converses of the implications from Corollary 15 are not valid. Indeed, for instance, if L is a Boolean algebra with more than three elements, then L has BLP (thus L has* Filt*–BLP and* Id*–BLP), but L is neither* Filt*–local, nor* Id*–local. Furthermore, if L is a direct product of at least two non–trivial chains, then L has BLP (thus L has* Filt*–BLP and* Id*–BLP), but L is neither* Filt*–local, nor* Id*–local.*

Proposition 16 *If L is* Filt*–local, then, for any $F \in$* Filt(L)*, $\mathcal{B}(L/F) = \{0/F, 1/F\}$. Dually, if L is* Id*–local, then, for any $I \in$* Id(L)*, $\mathcal{B}(L/I) = \{0/I, 1/I\}$.*

Proof: Assume that L is Filt–local, and let $F \in$ Filt(L). Then, by Corollary 13 and Corollary 15, $\mathcal{B}(L) = \{0, 1\}$ and F has BLP, thus $\mathcal{B}(L/F) = \mathcal{B}(L)/F = \{0/F, 1/F\}$.   □

Remark 21 *Notice that, in fact, if L is* Filt*–local, then, for any proper filter F of L, L/F is* Filt*–local, and, dually, the same holds for ideals. Many proofs can be given for these facts; they can be proven directly from the definitions, or from Corollary 14, or even by using Proposition 17 below.*

Notice, also, that Corollary 15 could have been obtained from Proposition 16.

Proposition 17 *The following are equivalent:*

1. *If L is* Filt*–local;*

2. *L is non–trivial, L has* Filt*–BLP and $\mathcal{B}(L) = \{0, 1\}$.*

   Dually, the following are equivalent:

- *L is* Id*–local;*

- *L is non–trivial, L has* Id*–BLP and $\mathcal{B}(L) = \{0, 1\}$.*

Proof: 1⇒2: By Corollary 15 and Corollary 13.
2⇒1: Assume that L is non–trivial and it has Filt–BLP and $\mathcal{B}(L) = \{0, 1\}$. Then, by Lemma 4, Proposition 12 and Lemma 5, we get that: L is B–conormal, hence, for all $x, y \in L$ with $x \wedge y = 0$, it follows that $x = x \wedge 1 = 0$ or $y = y \wedge 1 = 0$, thus L is Filt–local.   □

Remark 22 *Corollary 15 provides us with a quite productive method to obtain bounded distributive lattices with* Filt*–BLP and/or* Id*–BLP and bounded distributive lattices without* Filt*–BLP and/or* Id*–BLP. Let us first notice that the ordinal sum between a bounded lattice A and the trivial bounded lattice is A, and, similarly, the ordinal sum between the trivial bounded lattice and a bounded lattice B is B. Now let us analyze the following situations, in which the Hasse diagrams for the bounded distributive lattice L are suggested by the pictures below:*
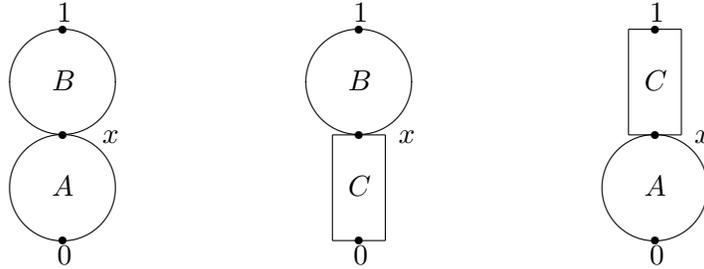
1. *if L is the ordinal sum between two non–trivial bounded distributive lattices A and B, then it is immediate, by Corollary 14, that: L is* Filt*–local iff A is* Filt*–local, and L is* Id*–local iff B is* Id*–local; hence, according to Corollary 15: if A is* Filt*–local, then L has* Filt*–BLP, and, if B is* Id*–local, then L has* Id*–BLP;*

2. *if L is the ordinal sum between a non–trivial bounded chain C and a bounded distributive lattice B, then, by 1, Remark 19 and Corollary 15, we get that L is* Filt*–local, hence L has* Filt*–BLP;*

3. *if L is the ordinal sum between a bounded distributive lattice A and a non–trivial bounded chain C, then, by 1, Remark 19 and Corollary 15, we get that L is* Id*–local, hence L has* Id*–BLP.*

For instance, see Remark 13 and notice that the bounded distributive lattice in Example 2 is the ordinal sum between the two–element chain and the rhombus, while the one in Example 1 is the ordinal sum between the rhombus and the two–element chain. The rhombus is a Boolean algebra, which has BLP, thus it has Filt*–BLP and* Id*–BLP, hence, if L is the ordinal sum between two bounded distributive lattices A and B, then: Example 1 shows that, if A has* Filt*–BLP, then L does not necessarily have* Filt*–BLP, while Example 2 shows that, if B has* Id*–BLP, then L does not necessarily have* Id*–BLP. Therefore these strengthenings of the implications from 1 above do not hold.*

Acknowledgements

References

[1] D. D. Anderson, V. P. Camillo. Commutative Rings Whose Elements Are a Sum of a Unit and an Idempotent. *Commutative Algebra* 30(7), 3327–3336, 2002. `doi:10.1081/AGB-120004490`.

[2] R. Balbes, P. Dwinger. *Distributive Lattices.* University of Missouri Press, Columbia, Missouri, 1974.

[3] G. Birkhoff. *Lattice Theory.* American Mathematical Society Colloquium Publications, Vol. 25, Providence, Rhode Island, Third Edition, 1973.

[4] T. S. Blyth. *Lattices and Ordered Algebraic Structures.* Springer–Verlag London Limited, 2005. `doi:10.1007/b139095`.

[5] A. Brezuleanu, R. Diaconescu. Sur la duale de la categorie des treillis. *Revue Roumaine de Mathématiques Pures et Appliquées* 14, 311–323, 1969.

[6] V. P. Camillo, H. P. Yu. Exchange Rings, Units and Idempotents. *Communications in Algebra* 22(12), 4737–4749, 1994. `doi:10.1080/00927879408825098`.

[7] D. Cheptea, G. Georgescu, C. Mureşan. Algebraic and Topological Results on Lifting Properties in Residuated Lattices (submitted).

[8] T. Coquand, H. Lombardi, C. Quitte. Dimension de Heitmann des Treillis Distributifs et des Anneaux Commutatifs. *Publications Mathématiques de Besançon. Algèbre et Théorie des Nombres*, 57–100, 2003–2006.

[9] A. Di Nola, G. Georgescu, L. Leuştean. Boolean Products of BL–algebras. *Journal of Mathematical Analysis and Applications* 251(1), 106–131, 2000. `doi:10.1006/jmaa.2000.7024`.

[10] A. Filipoiu, G. Georgescu, A. Lettieri. Maximal MV–algebras. *Mathware & Soft Computing* 4(1), 53–62, 1997.

[11] G. Georgescu, L. Leuştean, C. Mureşan. Maximal Residuated Lattices with Lifting Boolean Center. *Algebra Universalis* 63(1), 83–99, February 2010. `doi:10.1007/s00012-010-0066-3`.

[12] G. Georgescu, C. Mureşan. Boolean Lifting Property for Residuated Lattices. *Soft Computing* 18(11), 2075–2089, November 2014. `doi:10.1007/s00500-014-1318-5`.

[13] G. Grätzer. *General Lattice Theory*. Birkhäuser Akademie–Verlag, Basel–Boston–Berlin, 1978.

[14] B. Jónsson. Congruence Distributive Varieties. *Mathematica Japonica* 42(2), 353–401, 1995.

[15] P. T. Johnstone. *Stone Spaces. Cambridge Studies in Advanced Mathematics* 3, Cambridge University Press, Cambridge/London/New York/New Rochelle/Melbourne/Sydney, 1982.

[16] T. Y. Lam. *Lectures on Modules and Rings. Graduate Texts in Mathematics* 189, Springer–Verlag, 1999.

[17] L. Leuştean. *Representations of Many–valued Algebras.* Editura Universitară, Bucharest, 2010.

[18] C. Mureşan. *Algebras of Many–valued Logics. Contributions to the Theory of Residuated Lattices.* Ph. D. Thesis, University of Bucharest, Bucharest, 2010.

[19] C. Mureşan. Characterization of the Reticulation of a Residuated Lattice. *Journal of Multiple–valued Logic and Soft Computing* 16 (3–5), Special Issue: *Multiple–valued Logic and Its Algebras*, 427–447, 2010.

[20] C. Mureşan. Dense Elements and Classes of Residuated Lattices. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie* 53 (101), No. 1, 11–24, 2010.

[21] W. K. Nicholson. Lifting Idempotents and Exchange Rings. *Transactions of the American Mathematical Society* 229, 269–278, 1977. `doi:10.1090/S0002-9947-1977-0439876-2`.

[22] R. Padmanabhan, S. Rudeanu. *Axioms for Lattices and Boolean Algebras.* World Scientific Publ. Co., Hackensack, NJ, 2008.