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The Reticulation of a Universal Algebra

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Abstract

The *reticulation* of an algebra A is a bounded distributive lattice $\mathcal{L}(A)$ whose prime spectrum of filters or ideals is homeomorphic to the prime spectrum of congruences of A, endowed with the Stone topologies. We have obtained a construction for the reticulation of any algebra A from a semi–degenerate congruence–modular variety $\mathcal{C}$ in the case when the commutator of A, applied to compact congruences of A, produces compact congruences, in particular when $\mathcal{C}$ has principal commutators; furthermore, it turns out that weaker conditions than the above are sufficient for A to have a reticulation. This construction generalizes the reticulation of a commutative unitary ring, as well as that of a residuated lattice, which in turn generalizes the reticulation of a BL–algebra and that of an MV–algebra. The purpose of constructing the reticulation for the algebras from $\mathcal{C}$ is that of transferring algebraic and topological properties between the variety of bounded distributive lattices and $\mathcal{C}$, and a *reticulation functor* is particularily useful for this transfer. We have defined and studied a reticulation functor for our construction of the reticulation in this context of universal algebra.

Keywords: (congruence–modular, congruence–distributive) variety, commutator, (prime, compact) congruence, reticulation.

1 Introduction

The reticulation of a commutative unitary ring R is a bounded distributive lattice $\mathcal{L}(R)$ whose prime spectrum of ideals is homeomorphic to the prime spectrum of ideals of R. Its construction has appeared in [30], but it has

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been extensively studied in [49], where it has received the name *reticulation*. The mapping $R \mapsto \mathcal{L}(R)$ sets a covariant functor from the category of commutative unitary rings to that of bounded distributive lattices, through which properties can be transferred between these categories. In [8], the reticulation has been defined and studied for non–commutative unitary rings and it has been proven that such a ring has a reticulation (with the topological definition above) iff it is quasi–commutative.

Over the past two decades, reticulations have been constructed for orderred algebras related to logic: MV–algebras [7, 9], BL–algebras [35, 19, 36], residuated lattices [39, 40, 41, 42, 43, 44], 0–distributive lattices [47], almost distributive lattices [48], Hilbert algebras [13], hoops [15]. All these algebras posess a "prime spectrum" which is homeomorphic to the prime spectrum of filters or ideals of a bounded distributive lattice; their reticulations consist of such bounded distributive lattices, whose study involves obtaining a construction for them and using that construction to transfer properties between these classes of algebras and bounded distributive lattices.

The purpose of the present paper is to set the problem of constructing a reticulation in a universal algebra framework and providing a solution to this problem in a case as general as possible, that includes the cases of the varieties above and generalizes the constructions which have been obtained in those particular cases. Apart from the novelty of using commutator theory [17, 37] for the study of the reticulation, essentially, the tools needed for obtaining reticulations in this very general setting are quite similar to those which have been put to work for the classes of algebras above, and it turns out that many types of results that hold for their reticulations can be generalized to our setting. In order to obtain strong generalizations, we have worked with hypotheses as weak as possible; all our results in this paper hold for semi–degenerate congruence–modular varieties whose members have the sets of compact congruences closed with respect to the commutator, with just a few exceptions that necessitate, moreover, principal commutators.

The present paper is structured as follows: Section 2 presents the notations and basic results we use in what follows; Section 3 collects a set of results from commutator theory which we use in the sequel, along with the standard construction of the Stone topologies on prime spectra, specifically the prime spectrum of ideals of a bounded distributive lattice and the prime spectrum of congruences of a universal algebra whose commutator fulfills certain conditions. The results in the following sections that are not cited from other papers, or mentioned as being either known or simple to obtain, are new and original.

In Section 4, we construct the reticulation for universal algebras whose commutators fulfill certain conditions, prove that this construction has the desired topological property and obtain some related results.

In Section 5, we provide some examples of reticulations, study particular cases, such as the congruence–distributive case, show that our construction generalizes constructions for the reticulation which have been obtained for particular varieties, and prove that our construction preserves finite direct products of algebras without skew congruences.

In Section 6, we obtain some arithmetical properties on commutators that we need in what follows and study the behaviour of Boolean congruences with respect to the reticulation.

In Section 7, we define a reticulation functor; our definition is not ideal, as it only acts on surjections; extending it to all morphisms remains an open problem. In this final section, we also show that the reticulation preserves quotients, and that it is a Boolean lattice exactly in the case of hyperarchimedean algebras, which we also characterize by several other conditions on their reticulation. These characterizations serve as an example for the transfer of properties to and from the category of bounded distributive lattices which the reticulation makes possible.

We conclude our present paper with Section 8, in which we lay out the main contributions of this paper and some directions for future research. We intend to further pursue the study of the reticulation in this universal algebra setting and use it to transfer more properties between the variety of bounded distributive lattices and the kinds of varieties that allow a construction for the reticulation. A theme for a potentially extensive future study is characterizing those varieties with the property that the reticulations of their members cover the entire class of bounded distributive lattices.

2 Preliminaries

For a further study of the following notions from universal algebra, we refer the reader to [1, 12, 25, 32]. For the lattice–theoretical ones, we recommend [5, 11, 16, 24].

We shall denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. For any set M, $\mathcal{P}(M)$ shall be the set of the subsets of M, $id_M : M \to M$ shall be the identity map, and we shall denote by $\Delta_M = \{(x, x) \mid x \in M\}$ and $\nabla_M = M^2$. For any family $(M_i)_{i\in I}$ of sets, by $(a_i)_{i\in I} \in \prod_{i\in I} M_i$ we mean

$a_i \in M_i$ for all $i \in I$. For any sets M, N and any function $f : M \to N$, we shall denote by $\mathrm{Ker}(f)$ the kernel of f, by f the direct image of $f^2 = f \times f$ and by f^* the inverse image of f^2.

Throughout this paper, whenever there is no danger of confusion, any algebra shall be designated by its support set. All algebras shall be considerred non–empty; by *trivial algebra* we mean one–element algebra. For brevity, we shall denote by $A \cong B$ the fact that two algebras A and B of the same type are isomorphic.

For any lattice L, we shall denote by $\mathrm{Id}(L)$, $\mathrm{Filt}(L)$, $\mathrm{PId}(L)$, $\mathrm{PFilt}(L)$, $\mathrm{Max}_{\mathrm{Id}}(L)$, $\mathrm{Max}_{\mathrm{Filt}}(L)$, $\mathrm{Spec}_{\mathrm{Id}}(L)$ and $\mathrm{Spec}_{\mathrm{Filt}}(L)$ the sets of the ideals, filters, principal ideals, principal filters, maximal ideals, maximal filters, prime ideals and prime filters of L, respectively; for any $M \subseteq L$ and any $a \in L$, $(M]$, respectively $[M)$, shall denote the ideal, respectively the filter of L generated by M, and we shall also use the notations $(a] = (\{a\}]$ and $[a) = [\{a\})$; whenever we need to specify the lattice L, we shall denote $[M)_L$, $(M]_L$, $[a)_L$ and $(a]_L$ instead of $[M)$, $(M]$, $[a)$ and $(a]$, respectively. For any bounded lattice L, any $U \subseteq L$ and any $u \in L$, we shall denote by $\mathrm{Ann}(U)$ and $\mathrm{Ann}(u)$ the *annihilator* of U and that of u in L, respectively: $\mathrm{Ann}(U) = \{a \in L \mid (\forall\, x \in U)\,(a \wedge x = 0)\}$ and $\mathrm{Ann}(u) = \mathrm{Ann}(\{u\})$; it is immediate that, if L is distributive, then $\mathrm{Ann}(U) = \mathrm{Ann}((U]_L) \in \mathrm{Id}(L)$.

For any algebra A, $\mathrm{Con}(A)$, $\mathrm{Max}(A)$, $\mathrm{PCon}(A)$ and $\mathcal{K}(A)$ shall denote the sets of the congruences, maximal congruences, principal congruences and finitely generated congruences of A, respectively. Clearly, $\Delta_A \in \mathrm{PCon}(A) \subseteq \mathcal{K}(A)$ and $\mathcal{K}(A)$ is the set of the compact elements of the lattice $\mathrm{Con}(A)$. For any $X \subseteq A^2$ and any $a, b \in A$, $Cg_A(X)$ shall be the congruence of A generated by X and we shall denote by $Cg_A(a, b) = Cg_A(\{(a, b)\})$. For any $\theta \in \mathrm{Con}(A)$, $p_\theta : A \to A/\theta$ shall be the canonical surjective morphism; if X belongs to A, $\mathcal{P}(A)$ or $\mathcal{P}(A^2)$, then we denote by $X/\theta = p_\theta(X)$.

Throughout the rest of this paper, τ shall be a universal algebras signature, $\mathcal{C}$ an equational class of τ–algebras, A and B algebras from $\mathcal{C}$ and $f : A \to B$ a morphism in $\mathcal{C}$. Unless mentioned otherwise, by *morphism* we mean τ–morphism. For any term t over τ, we shall denote by t^A the operation on A associated to t.

Note that, if $\beta \in \mathrm{Con}(B)$, then $\mathrm{Ker}(f) = f^*(\Delta_B) \subseteq f^*(\beta) \in \mathrm{Con}(A)$, thus $f^*(\beta) \in [\mathrm{Ker}(f))$; also, $f(f^*(\beta)) = \beta \cap f(A^2)$, thus, if f is surjective, then $f(f^*(\beta)) = \beta$. If $\alpha \in [\mathrm{Ker}(f))$, then $f(\alpha) \in \mathrm{Con}(f(A))$ and $f^*(f(\alpha)) = \alpha$, so, if f is surjective, then $f(\alpha) \in \mathrm{Con}(B)$. Thus, for any $\alpha \in \mathrm{Con}(A)$, we have $f(\alpha \vee \mathrm{Ker}(f)) \in \mathrm{Con}(f(A))$, so, if f is surjective, then $f(\alpha \vee \mathrm{Ker}(f)) \in$

Con(B). By the above, $\alpha \mapsto f(\alpha)$ is a lattice isomorphism from $[\mathrm{Ker}(f))$ to Con($f(A)$), thus to Con(B) if f is surjective, having the corresponding restriction of f^* as inverse; hence, if f is surjective, then this map sets an order isomorphism from $\mathrm{Max}(A) \cap [\mathrm{Ker}(f))$ to Max(B). In particular, for any $\theta \in \mathrm{Con}(A)$, $\alpha \mapsto p_\theta(\alpha) = \alpha/\theta$ is a lattice isomorphism from $[\theta)$ to Con(A/θ), which sets an order isomorphism from $\mathrm{Max}(A) \cap [\theta)$ to Max(A/θ).

3 The Commutator and the Stone Topology

For a further study of the results on commutators in this section, we refer the reader to [1, 20, 32, 46]; for those on Stone topologies, see [1, 17, 18, 23, 45, 28]. Out of the various definitions for commutator operations on congruence lattices, we have chosen to work with the *term condition commutator*, from the following definition. Recall that, in algebras from congruence–modular varieties, all definitions for the commutator give the same commutator operation.

Definition 1 [37] *Let $\alpha, \beta \in \mathrm{Con}(A)$. For any $\mu \in \mathrm{Con}(A)$, by $C(\alpha, \beta; \mu)$ we denote the fact that the following condition holds: for all $n, k \in \mathbb{N}$ and any term t over τ of arity $n+k$, if $(a_i, b_i) \in \alpha$ for all $i \in \overline{1, n}$ and $(c_j, d_j) \in \beta$ for all $j \in \overline{1, k}$, then $(t^A(a_1, \ldots, a_n, c_1, \ldots, c_k), t^A(a_1, \ldots, a_n, d_1, \ldots, d_k)) \in \mu$ iff $(t^A(b_1, \ldots, b_n, c_1, \ldots, c_k), t^A(b_1, \ldots, b_n, d_1, \ldots, d_k)) \in \mu$. We denote by $[\alpha, \beta]_A = \bigcap\{\mu \in \mathrm{Con}(A) \mid C(\alpha, \beta; \mu)\}$; we call $[\alpha, \beta]_A$ the* commutator of α and β *in A. The operation $[\cdot, \cdot]_A : \mathrm{Con}(A) \times \mathrm{Con}(A) \to \mathrm{Con}(A)$ is called the* commutator of A.

Note that, for all $\alpha, \beta \in \mathrm{Con}(A)$, we have $C(\alpha, \beta; \nabla_A)$; since Con($A$) is a complete lattice, it follows that $[\alpha, \beta]_A \in \mathrm{Con}(A)$. Furthermore, according to [37, Lemma 4.4,(2)], for any family $(\mu_i)_{i \in I} \subseteq \mathrm{Con}(A)$, if $C(\alpha, \beta; \mu_i)$ for all $i \in I$, then $C(\alpha, \beta; \bigcap_{i \in I} \mu_i)$. Hence $C(\alpha, \beta; [\alpha, \beta]_A)$, and thus $[\alpha, \beta]_A = \min\{\mu \in \mathrm{Con}(A) \mid C(\alpha, \beta; \mu)\}$, which is exactly the definition of the commutator from [38].

Theorem 1 [20] *If $\mathcal{C}$ is congruence–modular, then, for each member M of $\mathcal{C}$, $[\cdot, \cdot]_M$ is the unique binary operation on Con(M) such that, for all $\alpha, \beta \in \mathrm{Con}(M)$, $[\alpha, \beta]_M = \min\{\mu \in \mathrm{Con}(M) \mid \mu \subseteq \alpha \cap \beta$ and, for any member N of $\mathcal{C}$ and any surjective morphism $h : M \to N$ in $\mathcal{C}$, $\mu \vee \mathrm{Ker}(h) = h^*([h(\alpha \vee \mathrm{Ker}(h)), h(\beta \vee \mathrm{Ker}(h))]_N)\}$.*

Theorem 2 [29] *If $\mathcal{C}$ is congruence–distributive, then, in each member of $\mathcal{C}$, the commutator coincides to the intersection of congruences.*

Remark 1 *By [37, Lemma 4.6,Lemma 4.7,Theorem 8.3], the commutator is smaller than the intersection and increasing in both arguments, that is, for all $\alpha, \beta, \phi, \psi \in \mathrm{Con}(A)$, we have $[\alpha, \beta]_A \subseteq \alpha \cap \beta$ and, if $\alpha \subseteq \beta$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_A \subseteq [\beta, \psi]_A$; if $\mathcal{C}$ is congruence–modular, then the commutator is also commutative and distributive in both arguments with respect to arbitrary joins, that is $[\alpha, \beta]_A = [\beta, \alpha]_A$ for all $\alpha, \beta \in \mathrm{Con}(A)$ and, for any non–empty families $(\alpha_i)_{i \in I} \subseteq \mathrm{Con}(A)$ and $(\beta_j)_{j \in J} \subseteq \mathrm{Con}(A)$, $[\bigvee_{i \in I} \alpha_i, \bigvee_{j \in J} \beta_j]_A = \bigvee_{i \in I} \bigvee_{j \in J} [\alpha_i, \beta_j]_A$.*

Obviously, if $[\cdot, \cdot]_A$ equals the intersection and it is distributive w.r.t. the join, the latter of which holds if $\mathcal{C}$ is congruence–modular, then A is congruence–distributive.

By Theorem 1, if $\mathcal{C}$ is congruence–modular, $\alpha, \beta, \theta \in \mathrm{Con}(A)$ and f is surjective, then $[f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]_B = f([\alpha, \beta]_A \vee \mathrm{Ker}(f))$, thus $[(\alpha \vee \theta)/\theta, (\beta \vee \theta)/\theta]_B = ([\alpha, \beta]_A \vee \theta)/\theta$, hence, if $\theta \subseteq [\alpha, \beta]_A$, then $[\alpha/\theta, \beta/\theta]_{A/\theta} = [\alpha, \beta]_A/\theta$.

For brevity, most of the times, we shall use the remarks in this paper without referencing them, and the same goes for the lemmas and propositions that state basic results.

Lemma 1 [20] *If $\mathcal{C}$ is congruence–modular and S is a subalgebra of A, then, for any $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha \cap S^2, \beta \cap S^2]_S \subseteq [\alpha, \beta]_A \cap S^2$.*

Proposition 1 [46, Theorem 5.17, p. 48] *Assume that $\mathcal{C}$ is congruence–modular, and let $n \in \mathbb{N}^*$, $M_1, \ldots, M_n$ be algebras from $\mathcal{C}$, $M = \prod_{i=1}^{n} M_i$ and, for all $i \in \overline{1, n}$, $\alpha_i, \beta_i \in \mathrm{Con}(M_i)$. Then: $[\prod_{i=1}^{n} \alpha_i, \prod_{i=1}^{n} \beta_i]_M = \prod_{i=1}^{n} [\alpha_i, \beta_i]_{M_i}$.*

Following [32], we say that $\mathcal{C}$ is *semi–degenerate* iff no non–trivial algebra in $\mathcal{C}$ has one–element subalgebras. For instance, the class of unitary rings and any class of bounded orderred structures is semi–degenerate.

Definition 2 [20] *Let ϕ be a proper congruence of A. Then ϕ is called a* prime congruence *of A iff, for all $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha, \beta]_A \subseteq \phi$ implies $\alpha \subseteq \phi$*

or $\beta \subseteq \phi$. ϕ *is called a* semiprime congruence *of* A *iff, for all* $\alpha \in \mathrm{Con}(A)$, $[\alpha, \alpha]_A \subseteq \phi$ *implies* $\alpha \subseteq \phi$.

The set of the prime congruences of A shall be denoted by $\mathrm{Spec}(A)$. $\mathrm{Spec}(A)$ is called the *(prime) spectrum* of A and $\mathrm{Max}(A)$ is called the *maximal spectrum* of A. Not all algebras have prime congruences; however, by the following lemma, every non–trivial algebra from a semi–degenerate congruence–modular variety has prime congruences.

Lemma 2 [1, Theorem 5.3] *If* $\mathcal{C}$ *is congruence–modular and semi–degenerate, then any proper congruence of* A *is included in a maximal congruence of* A*, and any maximal congruence of* A *is prime.*

Proposition 2 [32] $\mathcal{C}$ *is semi–degenerate iff, for all members* M *of* $\mathcal{C}$, $\nabla_M \in \mathcal{K}(M)$.

Proposition 3 [20, Theorem 8.5, p. 85] *If* $\mathcal{C}$ *is congruence–modular, then the following are equivalent:*

(i) *for any algebra* M *from* $\mathcal{C}$, $[\nabla_M, \nabla_M]_M = \nabla_M$;

(ii) *for any algebra* M *from* $\mathcal{C}$ *and any* $\theta \in \mathrm{Con}(M)$, $[\theta, \nabla_M]_M = \theta$;

(iii) $\mathcal{C}$ *has no skew congruences, that is, for any algebras* M *and* N *from* $\mathcal{C}$, $\mathrm{Con}(M \times N) = \{\theta \times \zeta \mid \theta \in \mathrm{Con}(M), \zeta \in \mathrm{Con}(N)\}$.

Lemma 3 *If* $\mathcal{C}$ *is either congruence–distributive or both congruence–modular and semi–degenerate, then* $\mathcal{C}$ *fulfills the equivalent conditions from Proposition 3.*

Proof: The congruence–distributive case is clear from Theorem 2. The other case is exactly [1, Lemma 5.2]. □

Lemma 4 [6, Lemma 1.11], [50, Proposition 1.2] *If* f *is surjective, then, for any* $a, b \in A$*, any* $X \subseteq A^2$*, any* $\theta \in \mathrm{Con}(A)$ *and any* $\alpha, \beta \in [\mathrm{Ker}(f))$:

(i) $f(\theta \vee \mathrm{Ker}(f)) = Cg_B(f(\theta))$;

$f(\alpha \vee \beta) = f(\alpha) \vee f(\beta)$;

(ii) $f(Cg_A(a, b) \vee \mathrm{Ker}(f)) = Cg_B(f(a), f(b))$;

$f(Cg_A(X) \vee \mathrm{Ker}(f)) = Cg_B(f(X))$;

(iii) $(Cg_A(a,b)\vee\theta)/\theta = Cg_{A/\theta}(a/\theta, b/\theta)$;
$(Cg_A(X)\vee\theta)/\theta = Cg_{A/\theta}(X/\theta)$.

We say that A has *principal commutators* iff, for all $\alpha,\beta\in \mathrm{PCon}(A)$, we have $[\alpha,\beta]_A\in \mathrm{PCon}(A)$, that is iff $\mathrm{PCon}(A)$ is closed with respect to the commutator of A. We say that $\mathcal{C}$ has *principal commutators* iff each member of $\mathcal{C}$ has principal commutators. Clearly, $\mathcal{K}(A)$ is closed with respect to finite joins, and, if A has principal commutators and $[\cdot,\cdot]_A$ is commutative and distributive w.r.t. the join (for instance if $\mathcal{C}$ is congruence–modular), then $\mathcal{K}(A)$ is also closed with respect to the commutator of A. If $\mathcal{C}$ is congruence–distributive, then, as shown by Theorem 2: $\mathcal{C}$ has principal commutators iff $\mathcal{C}$ has the principal intersection property (PIP), while $\mathcal{K}(M)$ is closed with respect to the commutator for each member M of $\mathcal{C}$ iff $\mathcal{C}$ has the compact intersection property (CIP). As a particular case of the property of $\mathcal{K}(A)$ stated above, if $\mathcal{C}$ is congruence–distributive and has the PIP, then $\mathcal{C}$ has the CIP. We say that $\mathcal{C}$ has *associative commutators* iff, for each member M of $\mathcal{C}$, the commutator of M is an associative binary operation on $\mathrm{Con}(M)$.

Example 1 *[1, 10, 23], [29, Theorem 2.8], [31, 34] As shown by Theorem 2, any congruence–distributive variety has associative commutators. The variety of commutative unitary rings is semi–degenerate, congruence–modular, with principal commutators and associative commutators, and it is not congruence–distributive. Out of the semi–degenerate congruence–distributive varieties with the CIP, we mention semi–degenerate filtral varieties. Out of the semi–degenerate congruence–distributive varieties with the PIP, we mention: bounded distributive lattices, residuated lattices (a variety which includes Gödel algebras, product algebras, MTL–algebras, BL–algebras, MV–algebras) and semi–degenerate discriminator varieties (out of which we mention Boolean algebras, n–valued Post algebras, n–valued Łukasiewicz algebras, n–valued MV–algebras, n–dimensional cylindric algebras, Gödel residuated lattices).*

Let L be a bounded distributive lattice. For any $I\in \mathrm{Id}(L)$ and any $a\in L$, we shall denote by $V_{\mathrm{Id},L}(I)=\mathrm{Spec}_{\mathrm{Id}}(L)\cap[I)$, $D_{\mathrm{Id},L}(I)=\mathrm{Spec}_{\mathrm{Id}}(L)\setminus V_{\mathrm{Id},L}(I)$, $V_{\mathrm{Id},L}(a)=V_{\mathrm{Id},L}((a])$ and $D_{\mathrm{Id},L}(a)=D_{\mathrm{Id},L}((a])$. By replacing $\mathrm{Spec}_{\mathrm{Id}}(L)$ with $\mathrm{Spec}_{\mathrm{Filt}}(L)$, in the same way we can define $V_{\mathrm{Filt},L}(F)$, $D_{\mathrm{Filt},L}(F)$, $V_{\mathrm{Filt},L}(a)$ and $D_{\mathrm{Filt},L}(a)$ for any $F\in\mathrm{Filt}(L)$ and any $a\in L$. It is well known and straightforward that, if L is non–trivial, then $\{D_{\mathrm{Id},L}(I) \mid I \in \mathrm{Id}(L)\}$ is a topology on $\mathrm{Spec}_{\mathrm{Id}}(L)$, called the *Stone topology*, having $\{D_{\mathrm{Id},L}(a) \mid a \in L\}$ as a basis, $\{V_{\mathrm{Id},L}(I) \mid I \in \mathrm{Id}(L)\}$ as the family of closed sets and $\{V_{\mathrm{Id},L}(a) \mid a \in L\}$ as a basis of closed sets. Since $\mathrm{Max}_{\mathrm{Id}}(L) \subseteq \mathrm{Spec}_{\mathrm{Id}}(L)$, $\{D_{\mathrm{Id},L}(I) \cap \mathrm{Max}_{\mathrm{Id}}(L) \mid I \in \mathrm{Id}(L)\}$ is a topology on $\mathrm{Max}_{\mathrm{Id}}(L)$, which is also called the *Stone topology*, and it has $\{D_{\mathrm{Id},L}(a) \cap \mathrm{Max}_{\mathrm{Id}}(L) \mid a \in L\}$ as a basis, $\{V_{\mathrm{Id},L}(I) \cap \mathrm{Max}_{\mathrm{Id}}(L) \mid I \in \mathrm{Id}(L)\}$ as the family of closed sets and $\{V_{\mathrm{Id},L}(a) \cap \mathrm{Max}_{\mathrm{Id}}(L) \mid a \in L\}$ as a basis of closed sets. Dually, we have the Stone topologies on $\mathrm{Spec}_{\mathrm{Filt}}(L)$ and $\mathrm{Max}_{\mathrm{Filt}}(L)$. $\mathrm{Spec}_{\mathrm{Id}}(L)$, $\mathrm{Max}_{\mathrm{Id}}(L)$, $\mathrm{Spec}_{\mathrm{Filt}}(L)$ and $\mathrm{Max}_{\mathrm{Filt}}(L)$ are called the *(prime) spectrum of ideals*, *maximal spectrum of ideals*, *(prime) spectrum of filters* and *maximal spectrum of filters* of L, respectively.

Throughout the rest of this section, we shall assume that $[\cdot,\cdot]_A$ is commutative and distributive w.r.t. arbitrary joins, which holds if $\mathcal{C}$ is congruence–modular. For each $\theta \in \mathrm{Con}(A)$, we shall denote by $V_A(\theta) = \mathrm{Spec}(A) \cap [\theta) = \{\phi \in \mathrm{Spec}(A) \mid \theta \subseteq \phi\}$ and by $D_A(\theta) = \mathrm{Spec}(A) \setminus V_A(\theta) = \{\psi \in \mathrm{Spec}(A) \mid \theta \nsubseteq \psi\}$. We shall also denote, for any $a, b \in A$, by $V_A(a,b) = V_A(Cg_A(a,b)) = \{\phi \in \mathrm{Spec}(A) \mid (a,b) \in \phi\}$ and by $D_A(a,b) = D_A(Cg_A(a,b)) = \{\psi \in \mathrm{Spec}(A) \mid (a,b) \notin \psi\}$. The proof of the following result is straightforward.

Proposition 4 [1] *If* $\mathrm{Spec}(A)$ *is non–empty, then* $(\mathrm{Spec}(A), \{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\})$ *is a topological space, having* $\{D_A(a,b) \mid a,b \in A\}$ *as a basis and in which, for all* $\alpha, \beta \in \mathrm{Con}(A)$ *and any family* $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$*, the following hold:*

- $D_A(\Delta_A) = \emptyset$ *and* $D_A(\nabla_A) = \mathrm{Spec}(A)$*;*
  $V_A(\Delta_A) = \mathrm{Spec}(A)$ *and* $V_A(\nabla_A) = \emptyset$*;*
- $D_A([\alpha,\beta]_A) = D_A(\alpha \cap \beta) = D_A(\alpha) \cap D_A(\beta)$*;*
  $V_A([\alpha,\beta]_A) = V_A(\alpha \cap \beta) = V_A(\alpha) \cup V_A(\beta)$*;*
- $D_A(\bigvee_{i\in I} \alpha_i) = \bigcup_{i\in I} D_A(\alpha_i)$*;*
  $V_A(\bigvee_{i\in I} \alpha_i) = \bigcap_{i\in I} V_A(\alpha_i)$*.*

Proposition 5 *If* $\mathcal{C}$ *is congruence–modular and semi–degenerate, then, for any* $\alpha \in \mathrm{Con}(A)$*:* $D_A(\alpha) = \mathrm{Spec}(A)$ *iff* $V_A(\alpha) = \emptyset$ *iff* $\alpha = \nabla_A$*.*

Proof: $D_A(\alpha) = \mathrm{Spec}(A)$ iff $\mathrm{Spec}(A) \setminus D_A(\alpha) = \emptyset$ iff $V_A(\alpha) = \emptyset$. Since $\mathrm{Spec}(A) \subseteq \mathrm{Con}(A) \setminus \{\nabla_A\}$, we have $V_A(\nabla_A) = \emptyset$, which was also part of Proposition 4. If $\alpha \neq \nabla_A$, then, according to Lemma 2, there exists a $\phi \in \mathrm{Spec}(A)$ such that $\alpha \subseteq \phi$, that is $V_A(\alpha) \neq \emptyset$. □

$\{D_A(\theta) \mid \theta \in \mathrm{Con}(A)\}$ is called the *Stone topology* on $\mathrm{Spec}(A)$. Obviously, its family of closed sets is $\{V_A(\theta) \mid \theta \in \mathrm{Con}(A)\}$, and $\{V_A(a, b) \mid a, b \in A\}$ is a basis of closed sets for this topology. The Stone topology on $\mathrm{Spec}(A)$ induces the *Stone topology* on $\mathrm{Max}(A)$, namely $\{D_A(\theta) \cap \mathrm{Max}(A) \mid \theta \in \mathrm{Con}(A)\}$.

Note that, if $\mathcal{C}$ is congruence–modular, B is a member of $\mathcal{C}$ and $f : A \to B$ is a surjective morphism, then the map $\alpha \mapsto f(\alpha)$ is an order isomorphism from $\mathrm{Spec}(A) \cap [\mathrm{Ker}(f))$ to $\mathrm{Spec}(B)$, and recall that the same goes for the maximal spectra. Hence, for any $\alpha \in [\mathrm{Ker}(f))$, we have $V_B(f(\alpha)) = f(V_A(\alpha))$ and $[f(\alpha)) \cap \mathrm{Max}(B) = f([\alpha) \cap \mathrm{Max}(A))$. In particular, for all $\theta \in \mathrm{Con}(A)$ and any $\alpha \in [\theta)$, we have $V_{A/\theta}(\alpha/\theta) = \{\psi/\theta \mid \psi \in V_A(\alpha)\}$ and $[\alpha/\theta) \cap \mathrm{Max}(A/\theta) = \{\psi/\theta \mid \psi \in [\alpha) \cap \mathrm{Max}(A)\}$.

4 The Construction of the Reticulation of a Universal Algebra and Related Results

Throughout this section, we shall assume that $[\cdot, \cdot]_A$ is commutative and distributive w.r.t. arbitrary joins, and that $\nabla_A \in \mathcal{K}(A)$, which hold in the particular case when $\mathcal{C}$ is congruence–modular and semi–degenerate. For every $\theta \in \mathrm{Con}(A)$, we shall denote by $\rho_A(\theta)$ the *radical* of θ, that is the intersection of the prime congruences of A which include θ: $\rho_A(\theta) = \bigcap\{\phi \in \mathrm{Spec}(A) \mid \theta \subseteq \phi\} = \bigcap_{\phi \in V_A(\theta)} \phi$.

Remark 2 *Let $\alpha, \beta \in \mathrm{Con}(A)$ and $\phi \in \mathrm{Spec}(A)$. Then, clearly, $V_A(\nabla_A) = \emptyset$, and thus $\rho_A(\nabla_A) = \nabla_A$. $\alpha \subseteq \rho_A(\alpha)$ and $\rho_A(\phi) = \phi$; moreover, $\rho_A(\alpha) = \alpha$ iff α is the intersection of a family of prime congruences of A. If $\alpha \subseteq \beta$, then $V_A(\alpha) \supseteq V_A(\beta)$, hence $\rho_A(\alpha) \subseteq \rho_A(\beta)$. Thus, if $\alpha \subseteq \phi$, then $\rho_A(\alpha) \subseteq \phi$.*

Following [1], for any $\alpha, \beta \in \mathrm{Con}(A)$ and every $n \in \mathbb{N}^*$, we denote by $[\alpha, \beta]^1_A = [\alpha, \beta]_A$ and $[\alpha, \beta]^{n+1}_A = [[\alpha, \beta]^n_A, [\alpha, \beta]^n_A]_A$, and by $(\alpha, \beta]^1_A = [\alpha, \beta]_A$ and $(\alpha, \beta]^{n+1}_A = (\alpha, (\alpha, \beta]^n_A]_A$.

Lemma 5 *For all $n \in \mathbb{N}^*$, any $\alpha, \beta \in \mathrm{Con}(A)$ and any family $(\alpha_i)_{i \in I} \in \mathrm{Con}(A)$:*

(i) $V_A(\alpha) = V_A(\rho_A(\alpha))$ *and*
$V_A(\bigvee_{i\in I} \alpha_i) = V_A(\bigvee_{i\in I} \rho_A(\alpha_i))$*;*

(ii) $V_A([\alpha,\beta]_A^n) = V_A([\alpha,\beta]_A) = V_A(\alpha\cap\beta) = V_A(\alpha)\cup V_A(\beta)$ *and*
$V_A([\alpha,\alpha]_A^n) = V_A([\alpha,\alpha]_A) = V_A(\alpha)$*.*

Proof: (i) $\alpha \subseteq \rho_A(\alpha)$, thus $V_A(\rho_A(\alpha)) \subseteq V_A(\alpha)$. If $\phi \in V_A(\alpha)$, then $\phi \in V_A(\rho_A(\alpha))$, so $V_A(\alpha) \subseteq V_A(\rho_A(\alpha))$. Thus $V_A(\alpha) = V_A(\rho_A(\alpha))$. Hence $V_A(\bigvee_{i\in I} \alpha_i) = \bigcap_{i\in I} V_A(\alpha_i) = \bigcap_{i\in I} V_A(\rho_A(\alpha_i)) = V_A(\bigvee_{i\in I} \rho_A(\alpha_i))$ by Proposition 4.
(ii) By Proposition 4, $V_A([\alpha,\beta]_A) = V_A(\alpha\cap\beta) = V_A(\alpha)\cup V_A(\beta)$. By induction on $n \in \mathbb{N}^*$, we prove that $V_A([\alpha,\beta]_A^n) = V_A(\alpha)\cup V_A(\beta)$. $V_A([\alpha,\beta]_A^1) = V_A([\alpha,\beta]_A) = V_A(\alpha)\cup V_A(\beta)$. Now let $n \in \mathbb{N}^*$ such that $V_A([\theta,\zeta]_A^n) = V_A(\theta)\cup V_A(\zeta)$ for all $\theta,\zeta \in \mathrm{Con}(A)$. Then $V_A([\alpha,\beta]_A^{n+1}) = V_A([[\alpha,\beta]_A^n, [\alpha,\beta]_A^n]_A) = V_A([\alpha,\beta]_A^n)\cup V_A([\alpha,\beta]_A^n) = V_A([\alpha,\beta]_A^n) = V_A(\alpha)\cup V_A(\beta)$. Hence $V_A([\alpha,\alpha]_A^n) = V_A([\alpha,\alpha]_A) = V_A(\alpha)$. □

Lemma 6 *For all* $\alpha,\beta,\theta \in \mathrm{Con}(A)$*, the following hold:*

(i) $\rho_A(\alpha) \subseteq \rho_A(\beta)$ *iff* $\alpha \subseteq \rho_A(\beta)$ *iff* $V_A(\alpha) \supseteq V_A(\beta)$*;*
$\rho_A(\alpha) = \rho_A(\beta)$ *iff* $V_A(\alpha) = V_A(\beta)$*;* $\rho_A(\rho_A(\alpha)) = \rho_A(\alpha)$*;*

(ii) if $\theta \subseteq \alpha$*, then* $\rho_{A/\theta}(\alpha/\theta) = \rho_A(\alpha)/\theta$*;* $\rho_{A/\theta}((\alpha\vee\theta)/\theta) = \rho_A(\alpha\vee\theta)/\theta$*;*
$\rho_{A/\theta}(\Delta_{A/\theta}) = \rho_A(\theta)/\theta$*.*

Proof: (i) Clearly, if $V_A(\alpha) \supseteq V_A(\beta)$, then $\rho_A(\alpha) \subseteq \rho_A(\beta)$. If $\rho_A(\alpha) \subseteq \rho_A(\beta)$, then, since $\alpha \subseteq \rho_A(\alpha)$, it follows that $\alpha \subseteq \rho_A(\beta)$. Finally, if $\alpha \subseteq \rho_A(\beta)$, then $V_A(\alpha) \supseteq V_A(\rho_A(\beta)) = V_A(\beta)$, by Lemma 5, (i). Therefore $\rho_A(\alpha) = \rho_A(\beta)$ iff $V_A(\alpha) = V_A(\beta)$. By Lemma 5, (i), it follows that $\rho_A(\rho_A(\alpha)) = \rho_A(\alpha)$.
(ii) If $\theta \subseteq \alpha$, then we may write: $\rho_{A/\theta}(\alpha/\theta) = \bigcap_{\psi\in V_{A/\theta}(\alpha/\theta)} \psi = \bigcap_{\phi\in V_A(\alpha)} \phi/\theta = (\bigcap_{\phi\in V_A(\alpha)} \phi)/\theta = \rho_A(\alpha)/\theta$. Thus $\rho_{A/\theta}((\alpha\vee\theta)/\theta) = \rho_A(\alpha\vee\theta)/\theta$ and $\rho_{A/\theta}(\Delta_{A/\theta}) = \rho_{A/\theta}(\theta/\theta) = \rho_A(\theta)/\theta$. □

Lemma 7 *For any* $n \in \mathbb{N}^*$*, any* $\alpha \in \mathrm{Con}(A)$ *and any family* $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$*:*

(i) $\rho_A([\alpha,\beta]^n_A)=\rho_A([\alpha,\beta]_A)=\rho_A(\alpha\cap\beta)=\rho_A(\alpha)\cap\rho_A(\beta)$*;*
$\rho_A([\alpha,\alpha]^n_A)=\rho_A([\alpha,\alpha]_A)=\rho_A(\alpha)$*;*

(ii) $\rho_A(\nabla_A)=\nabla_A$*;* $\rho_A(\bigvee_{i\in I}\rho_A(\alpha_i))=\rho_A(\bigvee_{i\in I}\alpha_i)$*;*

if $\mathcal{C}$ *is congruence–modular and semi–degenerate, then:*

$\rho_A(\alpha)=\nabla_A$ *iff* $\alpha=\nabla_A$*, and* $\bigvee_{i\in I}\rho_A(\alpha_i)=\nabla_A$ *iff* $\bigvee_{i\in I}\alpha_i=\nabla_A$*.*

Proof: (i) By Lemma 5, (ii), and Proposition 4, $\rho_A([\alpha,\beta]^n_A)=\rho_A([\alpha,\beta]_A)=\rho_A(\alpha\cap\beta)=\bigcap_{\phi\in V_A(\alpha\cap\beta)}\phi=\bigcap_{\phi\in V_A(\alpha)\cup V_A(\beta)}=\bigcap_{\phi\in V_A(\alpha)}\phi\cap\bigcap_{\phi\in V_A(\beta)}\phi=\rho_A(\alpha)\cap\rho_A(\beta)$. Hence $\rho_A([\alpha,\alpha]^n_A)=\rho_A([\alpha,\alpha]_A)=\rho_A(\alpha)$.
(ii) $\nabla_A\subseteq\rho_A(\nabla_A)$, thus $\rho_A(\nabla_A)=\nabla_A$. By Lemma 5, (i), $\rho_A(\bigvee_{i\in I}\rho_A(\alpha_i))=\rho_A(\bigvee_{i\in I}\alpha_i)$. If $\mathcal{C}$ is congruence–modular and semi–degenerate and $\alpha\neq\nabla_A$, then there exists $\phi\in V_A(\alpha)$, thus $\rho_A(\alpha)\subseteq\phi\subsetneq\nabla_A$; hence $\bigvee_{i\in I}\rho_A(\alpha_i)=\nabla_A$ iff $\rho_A(\bigvee_{i\in I}\rho_A(\alpha_i))=\nabla_A$ iff $\rho_A(\bigvee_{i\in I}\alpha_i)=\nabla_A$ iff $\bigvee_{i\in I}\alpha_i=\nabla_A$. □

The *radical congruences* of A are the congruences α of A such that $\alpha=\rho_A(\alpha)$. Let us denote by $\mathrm{RCon}(A)$ the set of the radical congruences of A. So $\mathrm{RCon}(A)=\{\alpha\in\mathrm{Con}(A)\mid\alpha=\rho_A(\alpha)\}=\{\rho_A(\alpha)\mid\alpha\in\mathrm{Con}(A)\}$, where the second equality follows from Lemma 6, (i). Clearly, $\mathrm{Spec}(A)\subseteq\mathrm{RCon}(A)$; moreover, the elements of $\mathrm{RCon}(A)$ are exactly the intersections of prime congruences of A.

Lemma 8 *If the commutator of* A *equals the intersection, in particular if* $\mathcal{C}$ *is congruence–distributive, then* $\mathrm{RCon}(A)=\mathrm{Con}(A)$.

Proof: By [1, Lemma 1.6], under the assumptions on the commutator at the beginning of this section, the radical congruences of A coincide to its semiprime congruences. Clearly, if $[\cdot,\cdot]_A=\cap$, then every congruence of A is semiprime, and thus radical. □

Most of the previous results on the radicals of congruences are known, but, for the sake of completeness, we have provided short proofs for them.

For any $\alpha, \beta \in \mathrm{Con}(A)$, let us denote by $\alpha \overset{\bullet}{\vee} \beta = \rho_A(\alpha \vee \beta)$. For any family $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$, we shall denote by $\overset{\bullet}{\bigvee_{i\in I}} \alpha_i = \rho_A(\bigvee_{i\in I} \alpha_i)$.

Proposition 6 $(\mathrm{RCon}(A), \overset{\bullet}{\vee}, \cap, \rho_A(\Delta_A), \rho_A(\nabla_A) = \nabla_A)$ *is a bounded lattice; moreover, it is a complete lattice, in which the arbitrary join is given by the* $\overset{\bullet}{\bigvee}$ *defined above.*

Proof: Clearly, $\overset{\bullet}{\vee}$ is commutative. By Lemma 7 and Lemma 6, (i), $\overset{\bullet}{\vee}$ is idempotent and associative, and, together with $\cap$, it fulfills the absorption laws. Therefore $(\mathrm{RCon}(A), \overset{\bullet}{\vee}, \cap)$ is a lattice, with $\rho_A(\Delta_A)$ as first element and $\rho_A(\nabla_A) = \nabla_A$ as last element. Now let us consider a family $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$, $M = \{\rho_A(\alpha_i) \mid i \in I\} \subseteq \mathrm{RCon}(A)$ and let us denote by $\theta = \overset{\bullet}{\bigvee_{i\in I}} \rho_A(\alpha_i) = \rho_A(\bigvee_{i\in I} \rho_A(\alpha_i)) = \rho_A(\bigvee_{i\in I} \alpha_i)$, by Lemma 7, (ii). Then $\theta \in \mathrm{RCon}(A)$ and $\rho_A(\alpha_i) \subseteq \theta$ for all $i \in I$. If $\zeta \in \mathrm{RCon}(A)$ and $\rho_A(\alpha_i) \subseteq \zeta$ for all $i \in I$, then $\bigvee_{i\in I} \rho_A(\alpha_i) \subseteq \zeta$, so, by Lemma 7, (ii), $\zeta = \rho_A(\zeta) \supseteq \rho_A(\bigvee_{i\in I} \rho_A(\alpha_i)) = \overset{\bullet}{\bigvee_{i\in I}} \rho_A(\alpha_i) = \theta$. Therefore $\theta = \sup(M)$ in the bounded lattice $\mathrm{RCon}(A)$, hence this lattice is complete. $\square$

Let us define a binary relation $\equiv_A$ on $\mathrm{Con}(A)$ by: $\alpha \equiv_A \beta$ iff $\rho_A(\alpha) = \rho_A(\beta)$, for any $\alpha, \beta \in \mathrm{Con}(A)$. $\equiv_A \cap (\mathcal{K}(A))^2$ shall also be denoted by $\equiv_A$. Clearly, $\equiv_A$ is an equivalence on $\mathrm{Con}(A)$, thus also on $\mathcal{K}(A)$. On $\mathrm{RCon}(A)$, $\equiv_A$ coincides to the equality, because, for any $\alpha, \beta \in \mathrm{Con}(A)$, $\rho_A(\alpha) \equiv_A \rho_A(\beta)$ iff $\rho_A(\rho_A(\alpha)) = \rho_A(\rho_A(\beta))$ iff $\rho_A(\alpha) = \rho_A(\beta)$. For all $\alpha \in \mathrm{Con}(A)$, let us denote by $\widehat{\alpha}$ the equivalence class of α with respect to $\equiv_A$; if $\alpha \in \mathcal{K}(A)$, then $\widehat{\alpha} \cap \mathcal{K}(A)$ will also be denoted by $\widehat{\alpha}$; let us also denote by $\mathbf{0} = \widehat{\Delta_A}$ and $\mathbf{1} = \widehat{\nabla_A}$. Let $\mathcal{L}(A) = \mathcal{K}(A)/_{\equiv_A} = \{\widehat{\theta} \cap \mathcal{K}(A) \mid \theta \in \mathcal{K}(A)\} = \{\widehat{\theta} \mid \theta \in \mathcal{K}(A)\}$, $\lambda_A : \mathrm{Con}(A) \to \mathrm{Con}(A)/_{\equiv_A}$ be the canonical surjection and let us denote in the same way the canonical surjection $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$.

Remark 3 *By Lemma 7, (ii), if $\mathcal{C}$ is congruence–modular and semi–degenerate, then, for any $\alpha \in \mathrm{Con}(A)$, $\widehat{\alpha} = \mathbf{1}$ iff $\rho_A(\alpha) = \nabla_A = \rho_A(\nabla_A)$ iff $\alpha = \nabla_A$.*

Lemma 9 $\equiv_A$ *is a congruence of the lattice* $\mathrm{Con}(A)$ *that also preserves the commutator, arbitrary joins and* $\dot{\bigvee}$ *over arbitrary families of congruences, and fulfills, for all* $\alpha, \beta \in \mathrm{Con}(A)$ *and all* $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$*:* $\alpha \equiv_A \rho_A(\alpha)$, $[\alpha, \beta]_A \equiv_A \alpha \cap \beta$ *and* $\dot{\bigvee_{i\in I}} \alpha_i \equiv_A \bigvee_{i\in I} \alpha_i$.

Proof: By Lemma 6, (i), $\alpha \equiv_A \rho_A(\alpha)$. By Lemma 7, (i), if $\alpha, \alpha', \beta, \beta' \in \mathrm{Con}(A)$ such that $\alpha \equiv_A \alpha'$ and $\beta \equiv_A \beta'$, then $[\alpha, \beta]_A \equiv_A [\alpha', \beta']_A \equiv_A \alpha \cap \beta \equiv_A \alpha' \cap \beta'$. By Lemma 6, (i), and Lemma 7, (ii), if $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$ and $(\alpha'_i)_{i\in I} \subseteq \mathrm{Con}(A)$ such that, for all $i \in I$, $\alpha_i \equiv_A \alpha'_i$, then $\dot{\bigvee_{i\in I}} \alpha_i \equiv_A \dot{\bigvee_{i\in I}} \alpha'_i \equiv_A \bigvee_{i\in I} \alpha_i \equiv_A \bigvee_{i\in I} \alpha'_i$. □

Proposition 7 $\mathrm{Con}(A)/\equiv_A$ *is a bounded distributive lattice, with first element* $\mathbf{0} = \widehat{\Delta_A} = \widehat{\rho_A(\Delta_A)}$ *and last element* $\mathbf{1} = \widehat{\nabla_A} = \widehat{\rho_A(\nabla_A)}$*, in which* $\widehat{\alpha} \wedge \widehat{\beta} = \widehat{[\alpha, \beta]_A}$ *for all* $\alpha, \beta \in \mathrm{Con}(A)$*, and* $\lambda_A : \mathrm{Con}(A) \to \mathrm{Con}(A)/\equiv_A$ *is a bounded lattice morphism. Moreover,* $\mathrm{Con}(A)/\equiv_A$ *is a frame, in which* $\bigvee_{i\in I} \widehat{\alpha_i} = \widehat{\bigvee_{i\in I} \alpha_i} = \widehat{\dot{\bigvee_{i\in I}} \alpha_i}$ *and* $\bigwedge_{i\in I} \widehat{\alpha_i} = \widehat{\bigcap_{i\in I} \alpha_i}$ *for any family* $(\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$.

Proof: By Lemma 9, $\equiv_A$ is a congruence of the bounded lattice $\mathrm{Con}(A)$ and $\alpha \equiv_A \rho_A(\alpha)$ for all $\alpha \in \mathrm{Con}(A)$, thus $\widehat{\Delta_A} = \widehat{\rho_A(\Delta_A)}$ and $(\mathrm{Con}(A)/\equiv_A, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a bounded lattice and the canonical surjection $\lambda_A : \mathrm{Con}(A) \to \mathrm{Con}(A)/\equiv_A$ is a bounded lattice morphism. The fact that the lattice $\mathrm{Con}(A)$ is complete and the surjectivity of the lattice morphism λ_A show that the lattice $\mathrm{Con}(A)/\equiv_A$ is complete and its joins and meets of arbitrary families of elements have the form in the enunciation, with the additional equality for joins following from Lemma 9. For any families $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$ of congruences of A, $(\bigvee_{i\in I} \widehat{\alpha_i}) \wedge (\bigvee_{j\in J} \widehat{\beta_j}) = ((\bigvee_{i\in I} \alpha_i) \cap (\bigvee_{j\in J} \beta_j))\widehat{\ } = ([\bigvee_{i\in I} \alpha_i, \bigvee_{j\in J} \beta_j]_A)\widehat{\ } = (\bigvee_{i\in I} \bigvee_{j\in J} [\alpha_i, \beta_j]_A)\widehat{\ } = \bigvee_{i\in I} \bigvee_{j\in J} \widehat{[\alpha_i, \beta_j]_A} = \bigvee_{i\in I} \bigvee_{j\in J} (\widehat{\alpha_i} \wedge \widehat{\beta_j})$, that is the meet is completely distributive with respect to the join in $\mathrm{Con}(A)/\equiv_A$, thus $\mathrm{Con}(A)/\equiv_A$ is a frame, in particular it is a bounded distributive lattice. □

We shall denote by $\leq$ the partial order of the lattice $\mathrm{Con}(A)/\equiv_A$.

Proposition 8 $(\mathrm{RCon}(A), \stackrel{\bullet}{\vee}, \cap, \rho_A(\Delta_A), \rho_A(\nabla_A) = \nabla_A)$ *is a frame, isomorphic to* $\mathrm{Con}(A)/\equiv_A$.

Proof: Let $\varphi : \mathrm{Con}(A)/\equiv_A \rightarrow \mathrm{RCon}(A)$, for all $\alpha \in \mathrm{Con}(A)$, $\varphi(\widehat{\alpha}) = \rho_A(\alpha)$. Then φ is surjective and, for all $\alpha, \beta \in \mathrm{Con}(A)$, the following equivalences hold: $\widehat{\alpha} = \widehat{\beta}$ iff $\alpha \equiv_A \beta$ iff $\rho_A(\alpha) = \rho_A(\beta)$ iff $\varphi(\widehat{\alpha}) = \varphi(\widehat{\beta})$, hence φ is well defined and injective. By Lemma 7, for all $\alpha, \beta \in \mathrm{Con}(A)$, $\varphi(\widehat{\alpha} \wedge \widehat{\beta}) = \varphi(\widehat{\alpha \cap \beta}) = \rho_A(\alpha \cap \beta) = \rho_A(\alpha) \cap \rho_A(\beta) = \varphi(\widehat{\alpha}) \cap \varphi(\widehat{\beta})$ and $\varphi(\widehat{\alpha} \vee \widehat{\beta}) = \varphi(\widehat{\alpha \vee \beta}) = \rho_A(\alpha \vee \beta) = \rho_A(\rho_A(\alpha) \vee \rho_A(\beta)) = \rho_A(\alpha) \stackrel{\bullet}{\vee} \rho_A(\beta) = \varphi(\widehat{\alpha}) \stackrel{\bullet}{\vee} \varphi(\widehat{\beta})$; actually, Lemma 7, (ii), and Proposition 7 show that φ preserves arbitrary joins. Therefore φ is a lattice isomorphism, thus an order isomorphism, hence it preserves arbitrary joins and meets. From this and Proposition 7 we obtain that $\mathrm{RCon}(A)$ is a frame and φ is a frame isomorphism. □

Throughout the rest of this section, we shall assume that $\mathcal{K}(A)$ is closed with respect to the commutator.

Proposition 9 $\mathcal{L}(A)$ *is a bounded sublattice of* $\mathrm{Con}(A)/\equiv_A$, *thus it is a bounded distributive lattice.*

Proof: Since $\nabla_A \in \mathcal{K}(A)$, we have $\mathbf{1} = \widehat{\nabla_A} \in \mathcal{L}(A)$. $\Delta_A \in \mathcal{K}(A)$, thus $\mathbf{0} = \widehat{\Delta_A} \in \mathcal{L}(A)$. Since $\mathcal{K}(A)$ is closed with respect to the commutator, for each $\alpha, \beta \in \mathcal{K}(A)$, we have $[\alpha, \beta]_A \in \mathcal{K}(A)$, thus $\widehat{\alpha} \wedge \widehat{\beta} = \widehat{[\alpha, \beta]_A} \in \mathcal{L}(A)$. For each $\alpha, \beta \in \mathcal{K}(A)$, $\widehat{\alpha} \vee \widehat{\beta} = \widehat{\alpha \vee \beta} \in \mathcal{L}(A)$. Hence $\mathcal{L}(A)$ is a bounded sublattice of $\mathrm{Con}(A)/\equiv_A$, which is distributive by Proposition 7, thus $\mathcal{L}(A)$ is a bounded distributive lattice. □

For any $\theta \in \mathrm{Con}(A)$ and any $I \in \mathrm{Id}(\mathcal{L}(A))$, we shall denote by:

- $\theta^* = \{\widehat{\alpha} \mid \alpha \in \mathcal{K}(A), \alpha \subseteq \theta\} = \lambda_A(\mathcal{K}(A) \cap (\theta]) \subseteq \mathcal{L}(A)$, where $(\theta] = (\theta]_{\mathrm{Con}(A)} \in \mathrm{PId}(\mathrm{Con}(A))$;
- $I_* = \bigvee\{\alpha \in \mathcal{K}(A) \mid \widehat{\alpha} \in I\} = \bigvee_{\alpha \in \lambda_A^{-1}(I)} \alpha \in \mathrm{Con}(A)$.

Since $\Delta_A \in \mathcal{K}(A)$ and thus $\widehat{\Delta_A} = \widehat{\rho_A(\Delta_A)} = \mathbf{0} \in I$, it follows that $\lambda_A^{-1}(I)$ is non–empty for any $I \in \mathrm{Id}(\mathcal{L}(A))$.

Lemma 10 *For all* $\theta \in \mathrm{Con}(A)$, $\theta^* \subseteq (\widehat{\theta}]_{\mathrm{Con}(A)/\equiv_A} \cap \mathcal{L}(A)$ *and* $\theta^* \in \mathrm{Id}(\mathcal{L}(A))$. *If* $\theta \in \mathcal{K}(A)$, *then* $\theta^* = (\widehat{\theta}]_{\mathrm{Con}(A)/\equiv_A} \cap \mathcal{L}(A) = (\widehat{\theta}]_{\mathcal{L}(A)} \in \mathrm{PId}(\mathcal{L}(A))$.

Proof: Let $\theta \in \mathrm{Con}(A)$, and, in this proof, let us denote by $\langle\widehat{\theta}\rangle = (\widehat{\theta}]_{\mathrm{Con}(A)/\equiv_A}$ and, in the case when $\theta \in \mathcal{K}(A)$, by $(\widehat{\theta}] = (\widehat{\theta}]_{\mathcal{L}(A)}$. $\theta^* = \{\widehat{\alpha} \mid \alpha \in (\theta] \cap \mathcal{K}(A)\}$.

For all $\alpha \in (\theta] \cap \mathcal{K}(A)$, we have $\widehat{\alpha} \in \mathcal{L}(A)$ and $\alpha \subseteq \theta$, thus $\widehat{\alpha} \leq \widehat{\theta}$ in $\mathrm{Con}(A)/\equiv_A$, hence $\widehat{\alpha} \in \langle\widehat{\theta}\rangle \cap \mathcal{L}(A)$, therefore $\theta^* \subseteq \langle\widehat{\theta}\rangle \cap \mathcal{L}(A)$. $\Delta_A \in \mathcal{K}(A)$ and $\Delta_A \subseteq \theta$, thus $\widehat{\Delta_A} \in \theta^*$, so θ^* is non–empty. Since $\mathcal{K}(A)$ is closed w.r.t. $[\cdot,\cdot]_A$, $\alpha \vee \beta, [\alpha,\beta]_A \in \mathcal{K}(A)$ for any $\alpha,\beta \in \mathcal{K}(A)$. Let $x, y \in \theta^*$, which means that $x = \widehat{\alpha}$ and $y = \widehat{\beta}$ for some $\alpha,\beta \in \mathcal{K}(A) \cap (\theta]$. Then $\alpha \vee \beta \in \mathcal{K}(A) \cap (\theta]$, thus $x \vee y = \widehat{\alpha} \vee \widehat{\beta} = \widehat{\alpha \vee \beta} \in \theta^*$. Now let $x \in \theta^*$ and $y \in \mathcal{L}(A)$ such that $x \geq y$, so that $y = x \wedge y$. Then $x = \widehat{\alpha}$ for some $\alpha \in \mathcal{K}(A) \cap (\theta]$ and $y = \widehat{\beta}$ for some $\beta \in \mathcal{K}(A)$. Thus $[\alpha,\beta]_A \in \mathcal{K}(A)$ and $[\alpha,\beta]_A \subseteq \alpha \cap \beta \subseteq \alpha \subseteq \theta$, hence $[\alpha,\beta]_A \in \mathcal{K}(A) \cap (\theta]$, therefore $y = x \wedge y = \widehat{\alpha} \wedge \widehat{\beta} = \widehat{[\alpha,\beta]_A} \in \theta^*$. Hence $\theta^* \in \mathrm{Id}(\mathcal{L}(A))$.

Now assume that $\theta \in \mathcal{K}(A)$, so that $\widehat{\theta} \in \mathcal{L}(A)$. By the above, $\theta^* \subseteq \langle\widehat{\theta}\rangle \cap \mathcal{L}(A) = (\widehat{\theta}]$. Let $x \in (\widehat{\theta}]$, so that there exists an $\alpha \in \mathcal{K}(A)$ with $\widehat{\alpha} = x \leq \widehat{\theta}$, thus $\widehat{[\alpha,\theta]_A} = \widehat{\alpha} \cap \widehat{\theta} = \widehat{\alpha} = x$. But $[\alpha,\theta]_A \in \mathcal{K}(A) \cap (\theta]$, so $x = \widehat{[\alpha,\theta]_A} \in \theta^*$. Therefore we also have $(\widehat{\theta}] \subseteq \theta^*$, hence $\theta^* = (\widehat{\theta}] \in \mathrm{PId}(\mathcal{L}(A))$. □

By the above, we have two functions: $\theta \in \mathrm{Con}(A) \mapsto \theta^* \in \mathrm{Id}(\mathcal{L}(A))$ and $I \in \mathrm{Id}(\mathcal{L}(A)) \mapsto I_* \in \mathrm{Con}(A)$.

Lemma 11 *(i) The two functions above are order–preserving.*

(ii) For any $\alpha \in \mathcal{K}(A)$ and any $I \in \mathrm{Id}(\mathcal{L}(A))$: $\alpha \subseteq I_*$ iff $\widehat{\alpha} \in I$.

Proof: (i) For any $\theta, \zeta \in \mathrm{Con}(A)$ such that $\theta \subseteq \zeta$, we have $(\theta] \subseteq (\zeta]$, hence $\theta^* \subseteq \zeta^*$. For any $I, J \in \mathrm{Id}(\mathcal{L}(A))$ such that $I \subseteq J$, we have $\lambda_A^{-1}(I) \subseteq \lambda_A^{-1}(J)$, thus $I_* \subseteq J_*$.
(ii) "$\Leftarrow$:" If $\widehat{\alpha} \in I$, then $\alpha \in \lambda_A^{-1}(I)$, thus $\alpha \subseteq I_*$.
"$\Rightarrow$:" If $\alpha \subseteq I_* = \bigvee\{\beta \in \mathcal{K}(A) \mid \widehat{\beta} \in I\}$, then, since $\alpha \in \mathcal{K}(A)$, it follows that there exist an $n \in \mathbb{N}^*$ and $\beta_1, \ldots, \beta_n \in \mathcal{K}(A)$ such that $\widehat{\beta_1}, \ldots, \widehat{\beta_n} \in I$ and $\alpha \subseteq \bigvee_{i=1}^{n} \beta_i$, hence $\widehat{\alpha} \subseteq \widehat{\bigvee_{i=1}^{n} \beta_i} = \bigvee_{i=1}^{n} \widehat{\beta_i} \in I$, thus $\widehat{\alpha} \in I$. □

Lemma 12 *(i) For any $\theta \in \mathrm{Con}(A)$, $\theta \subseteq (\theta^*)_*$.*

For any $\phi \in \mathrm{Spec}(A)$, $\phi = (\phi^*)_*$.

(ii) For any $I \in \mathrm{Id}(\mathcal{L}(A))$, $I = (I_*)^*$.

Proof: (i) For any $(a,b)\in\theta$, $Cg_A(a,b)\in \mathrm{PCon}(A)\subseteq \mathcal{K}(A)$ and $Cg_A(a,b)\subseteq\theta$, thus $Cg_A(a,b)\in\mathcal{K}(A)\cap(\theta]$, hence $\widehat{Cg_A(a,b)}\in\theta^*$, therefore $Cg_A(a,b)\subseteq(\theta^*)_*$ by Lemma 10 and Lemma 11, (ii), so $(a,b)\in(\theta^*)_*$. Hence $\theta\subseteq(\theta^*)_*$.

Thus $\phi\subseteq(\phi^*)_*$. Now let $\beta\in\mathcal{K}(A)$ such that $\widehat{\beta}\in\phi^*=\{\widehat{\alpha}\mid\alpha\in\mathcal{K}(A),\alpha\subseteq\phi\}$, which means that $\widehat{\beta}=\widehat{\alpha}$ for some $\alpha\in\mathcal{K}(A)$ with $\alpha\subseteq\phi$. Since $\widehat{\beta}=\widehat{\alpha}$, we have $\rho_A(\beta)=\rho_A(\alpha)$, while $\alpha\subseteq\phi$ gives us $\rho_A(\alpha)\subseteq\rho_A(\phi)=\phi$, where the last equality follows from the fact that $\phi\in\mathrm{Spec}(A)$. Hence $\beta\subseteq\rho_A(\beta)\subseteq\phi$. Therefore $(\phi^*)_*=\bigvee\{\gamma\in\mathcal{K}(A)\mid\widehat{\gamma}\in\phi^*\}\subseteq\phi$. Hence $\phi=(\phi^*)_*$.
(ii) For any $x\in\mathcal{L}(A)$, by Lemma 11, (ii), the following equivalences hold: $x\in(I_*)^*$ iff there exists an $\alpha\in\mathcal{K}(A)$ such that $\alpha\subseteq I_*$ and $x=\widehat{\alpha}$ iff there exists an $\alpha\in\mathcal{K}(A)$ such that $\widehat{\alpha}\in I$ and $x=\widehat{\alpha}$ iff $x\in I$. Therefore $(I_*)^*=I$. □

Proposition 10 *(i) The map $I\in\mathrm{Id}(\mathcal{L}(A))\mapsto I_*\in\mathrm{Con}(A)$ is injective.*

(ii) The map $\theta\in\mathrm{Con}(A)\mapsto\theta^*\in\mathrm{Id}(\mathcal{L}(A))$ is surjective.

Proof: (i) Let $I,J\in\mathrm{Id}(\mathcal{L}(A))$ such that $I_*=J_*$. Then $(I_*)^*=(J_*)^*$, so $I=J$ by Lemma 12, (ii).
(ii) Let $I\in\mathrm{Id}(\mathcal{L}(A))$, and denote $\theta=I_*\in\mathrm{Con}(A)$. Then $\theta^*=(I_*)^*=I$ by Lemma 12, (ii). □

Lemma 13 *(i) For any $\phi\in\mathrm{Spec}(A)$, we have $\phi^*\in\mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$.*

(ii) For any $P\in\mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$, we have $P_*\in\mathrm{Spec}(A)$.

Proof: (i) We have $\phi^*\in\mathrm{Id}(\mathcal{L}(A))=\mathrm{Id}(\mathcal{K}(A)/{\equiv_A})$. Let $\alpha,\beta\in\mathcal{K}(A)$ such that $\widehat{[\alpha,\beta]_A}=\widehat{\alpha}\wedge\widehat{\beta}\in\phi^*=\{\widehat{\gamma}\mid\gamma\in\mathcal{K}(A),\gamma\subseteq\phi\}$. Then there exists a $\gamma\in\mathcal{K}(A)$ such that $\gamma\subseteq\phi$ and $\widehat{\gamma}=\widehat{[\alpha,\beta]_A}$, thus $\rho_A(\gamma)=\rho_A([\alpha,\beta]_A)$ and $\rho_A(\gamma)\subseteq\rho_A(\phi)=\phi$ since $\phi\in\mathrm{Spec}(A)$. Hence $[\alpha,\beta]_A\subseteq\rho_A([\alpha,\beta]_A)\subseteq\phi$, hence $\alpha\subseteq\phi$ or $\beta\subseteq\phi$ since $\phi\in\mathrm{Spec}(A)$. But this means that $\widehat{\alpha}\in\phi^*$ or $\widehat{\beta}\in\phi^*$. Therefore $\phi^*\in\mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$.
(ii) We have $P_*\in\mathrm{Con}(A)$. Let $\alpha,\beta\in\mathrm{PCon}(A)$ such that $[\alpha,\beta]_A\subseteq P_*$. Then $\alpha,\beta\in\mathcal{K}(A)$, so that $[\alpha,\beta]_A\in\mathcal{K}(A)$, and $[\alpha,\beta]_A\subseteq\bigvee\{\gamma\in\mathcal{K}(A)\mid\widehat{\gamma}\in P\}$, hence there exist an $n\in\mathbb{N}^*$ and $\gamma_1,\ldots,\gamma_n\in\mathcal{K}(A)$ such that $\widehat{\gamma_1},\ldots,\widehat{\gamma_n}\in P$ and $[\alpha,\beta]_A\subseteq\bigvee_{i=1}^n\gamma_i$. But then $\widehat{\bigvee_{i=1}^n\gamma_i}=\bigvee_{i=1}^n\widehat{\gamma_i}\in P$, hence $\widehat{\alpha}\wedge\widehat{\beta}=\widehat{[\alpha,\beta]_A}\in P$, thus $\widehat{\alpha}\in P$ or $\widehat{\beta}\in P$ since $P\in\mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$. By Lemma 11, (ii), it follows that $\alpha\subseteq P_*$ or $\beta\subseteq P_*$. Therefore $P_*\in\mathrm{Spec}(A)$. □

By Lemma 13, we have these restrictions of the functions defined above:

- $u : \mathrm{Spec}(A) \to \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$, for all $\phi \in \mathrm{Spec}(A)$, $u(\phi) = \phi^*$;
- $v : \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A)) \to \mathrm{Spec}(A)$, for all $P \in \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$, $v(P) = P_*$.

Proposition 11 *u and v are homeomorphisms, inverses of each other, between the prime spectrum of A and the prime spectrum of ideals of $\mathcal{L}(A)$, endowed with the Stone topologies.*

Proof: By Lemma 12, (ii), for all $P \in \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$, we have $u(v(P)) = P$. By Lemma 12, (i), for all $\phi \in \mathrm{Spec}(A)$, we have $v(u(\phi)) = \phi$. Thus u and v are bijections and they are inverses of each other.

Let $\theta \in \mathrm{Con}(A)$ and $\phi \in V_A(\theta)$, that is $\phi \in \mathrm{Spec}(A)$ and $\theta \subseteq \phi$. Then, by Lemma 13, (ii), and Lemma 11, (i), $\phi^* \in \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$ and $\theta^* \subseteq \phi^*$, so $\phi^* \in V_{\mathrm{Id},\mathcal{L}(A)}(\theta^*)$, and we have $u(\phi) = \phi^*$. Hence $u(V_A(\theta)) \subseteq V_{\mathrm{Id},\mathcal{L}(A)}(\theta^*)$. Now let $P \in V_{\mathrm{Id},\mathcal{L}(A)}(\theta^*)$, that is $P \in \mathrm{Spec}_{\mathrm{Id}}(\mathcal{L}(A))$ and $\theta^* \subseteq P$. Then, by Lemma 12, (i), Lemma 11, (i), and Lemma 13, (ii), $\theta \subseteq (\theta^*)_* \subseteq P_* \in \mathrm{Spec}(A)$, thus $P_* \in V_A(\theta)$, and we have $u(P_*) = u(v(P)) = P$. Hence $V_{\mathrm{Id},\mathcal{L}(A)}(\theta^*) \subseteq u(V_A(\theta))$. Therefore $u(V_A(\theta)) = V_{\mathrm{Id},\mathcal{L}(A)}(\theta^*)$, thus u is closed, hence u is open, so v is continuous.

Now let $I \in \mathrm{Id}(\mathcal{L}(A))$. Then, according to Proposition 10, (ii), $I = \theta^*$ for some $\theta \in \mathrm{Con}(A)$. By the above, $u(V_A(\theta)) = V_{\mathrm{Id},\mathcal{L}(A)}(\theta^*) = V_{\mathrm{Id},\mathcal{L}(A)}(I)$, hence $v(V_{\mathrm{Id},\mathcal{L}(A)}(I)) = v(u(V_A(\theta))) = V_A(\theta)$, therefore v is closed, hence v is open, thus u is continuous.

Hence u and v are homeomorphisms. □

Proposition 12 [5, 24] *If L and M are bounded distributive lattices whose prime spectra of ideals, endowed with the Stone topologies, are homeomorphic, then L and M are isomorphic.*

Theorem 3 *(i) $\mathcal{L}(A)$ is a reticulation for the algebra A.*

(ii) The reticulation of A is unique up to a lattice isomorphism.

Proof: (i) By Propositions 11 and 9.
(ii) By Proposition 12. □

Corollary 1 *If $\mathcal{C}$ is congruence–modular and semi–degenerate, then u and v induce homeomorphisms, inverses of each other, between the maximal spectrum of A and the maximal spectrum of ideals of $\mathcal{L}(A)$, endowed with the Stone topologies.*

Proof: By Lemma 2, Proposition 11 and the fact that, as Lemma 11, (i), ensures us, u and v are order–preserving, and hence they are order isomorphisms between the posets $(\text{Spec}(A), \subseteq)$ and $(\text{Spec}_{\text{Id}}(\mathcal{L}(A)), \subseteq)$. □

Proposition 13 [1, Proposition 4.1] *For any* $\theta \in \text{Con}(A)$, $\rho_A(\theta) = \{(a, b) \in A^2 \mid (\exists\, n \in \mathbb{N}^*)\,([Cg_A(a, b), Cg_A(a, b)]^n_A \subseteq \theta)\}$, *so* $\rho_A(\Delta_A) = \{(a, b) \in A^2 \mid (\exists\, n \in \mathbb{N}^*)\,([Cg_A(a, b), Cg_A(a, b)]^n_A = \Delta_A)\}$.

Lemma 14 *(i) For all* $\theta \in \text{Con}(A)$, $(\theta^*)_* = \rho_A(\theta)$ *and* $\rho_A(\theta)^* = \theta^*$.

(ii) For all $I \in \text{Id}(\mathcal{L}(A))$, $\rho_A(I_*) = I_*$.

Proof: (i) For every $\beta \in \mathcal{K}(A)$ such that $\widehat{\beta} \in \theta^* = \{\widehat{\gamma} \mid \gamma \in \mathcal{K}(A), \gamma \subseteq \theta\}$, there exists an $\alpha \in \mathcal{K}(A)$ such that $\alpha \subseteq \theta$ and $\widehat{\alpha} = \widehat{\beta}$, thus $\beta \subseteq \rho_A(\beta) = \rho_A(\alpha) \subseteq \rho_A(\theta)$. Therefore $(\theta^*)_* = \bigvee\{\gamma \in \mathcal{K}(A) \mid \widehat{\gamma} \in \theta^*\} \subseteq \rho_A(\theta)$. Now let $(a, b) \in \rho_A(\theta)$, so that, according to Proposition 13, Lemma 12, (i), and Lemma 11, (ii), for some $n \in \mathbb{N}^*$, $[Cg_A(a, b), Cg_A(a, b)]^n_A \subseteq \theta \subseteq (\theta^*)_*$, hence $([Cg_A(a, b), Cg_A(a, b)]^n_A)\widehat{\ } \in \theta^*$. But $\rho_A([Cg_A(a, b), Cg_A(a, b)]^n_A) = \rho_A(Cg_A(a, b))$, thus $\widehat{Cg_A(a, b)} = ([Cg_A(a, b), Cg_A(a, b)]^n_A)\widehat{\ } \in \theta^*$, hence $(a, b) \in Cg_A(a, b) \subseteq (\theta^*)_*$ by Lemma 11, (ii). Therefore $\rho_A(\theta) \subseteq (\theta^*)_*$. Hence $(\theta^*)_* = \rho_A(\theta)$. By Lemma 12, (ii), it follows that $\theta^* = ((\theta^*)_*)^* = \rho_A(\theta)^*$.
(ii) By (i) and Lemma 12, (ii), we have $\rho_A(I_*) = ((I_*)^*)_* = I_*$. □

Proposition 14 *The maps* $\theta \in \text{RCon}(A) \mapsto \theta^* \in \text{Id}(\mathcal{L}(A))$ *and* $I \in \text{Id}(\mathcal{L}(A)) \mapsto I_* \in \text{RCon}(A)$ *are frame isomorphisms and inverses of each other.*

Proof: By Lemma 14, (ii), for all $I \in \text{Id}(\mathcal{L}(A))$, we have $I_* \in \text{RCon}(A)$, hence the second map above is well defined. By Lemma 12, (ii), for all $I \in \text{Id}(\mathcal{L}(A))$, $(I_*)^* = I$. By Lemma 14, (i), for all $\theta \in \text{RCon}(A)$, $\theta = \rho_A(\theta) = (\theta^*)_*$. Hence these functions are inverses of each other, thus they are bijections. By Lemma 11, (i), these maps are order–preserving, thus they are order isomorphisms, hence they preserve arbitrary joins and meets, therefore they are frame isomorphisms. □

5 Some Examples, Particular Cases and Preservation of Finite Direct Products

Throughout this section, we shall assume that $[\cdot, \cdot]_A$ is commutative and distributive w.r.t. arbitrary joins and $\nabla_A \in \mathcal{K}(A)$. These hypotheses are sufficient for the following results we cite from other works to hold, and the complete tables of the commutators for the following algebras show that their commutators fulfill these properties. We shall denote by $\mathcal{HSP}(A)$ the variety generated by A. By [1, Proposition 1.2], if $\mathcal{HSP}(A)$ is congruence–modular, then, for each proper congruence ϕ of A: ϕ is prime iff ϕ is semiprime and meet–irreducible in the lattice $\mathrm{Con}(A)$. Of course, since each of the algebras M from the following examples is finite, we have $\nabla_M \in \mathcal{K}(M)$. We have used the method in [38] to calculate the commutators, excepting those in groups, where we have used the commutators on normal subgroups; recall that the variety of groups is congruence–modular [37]. Following [1], we say that A is: *Abelian* iff $[\nabla_A, \nabla_A]_A = \Delta_A$, *solvable* iff $[\nabla_A, \nabla_A]_A^n = \Delta_A$ for some $n \in \mathbb{N}^*$, *nilpotent* iff $(\nabla_A, \nabla_A]_A^n = \Delta_A$ for some $n \in \mathbb{N}^*$. For any $n \in \mathbb{N}^*$, we shall denote by $\mathcal{L}_n$ the n–element chain. By $\oplus$ we shall denote the ordinal sum of bounded lattices.

Remark 4 *If* $\mathrm{Spec}(A) = \emptyset$*, then* $\rho_A(\alpha) = \nabla_A$ *for all* $\alpha \in \mathrm{Con}(A)$*, hence* $\equiv_A = \nabla_{\mathcal{K}(A)}$*, thus* $\mathcal{L}(A) = \mathcal{K}(A)/\nabla_{\mathcal{K}(A)} \cong \mathcal{L}_1$*. If* $\mathrm{Spec}(A) = \{\phi\}$ *for some* $\phi \in \mathrm{Con}(A) \setminus \{\nabla_A\}$*, then:* $\rho_A(\theta) = \phi = \rho_A(\Delta_A)$ *for all* $\theta \in (\phi]$*, and* $\rho_A(\theta) = \nabla_A = \rho_A(\nabla_A)$ *for all* $\theta \in \mathrm{Con}(A) \setminus (\phi]$*, therefore, since* $\Delta_A, \nabla_A \in \mathcal{K}(A)$*,* $\mathcal{L}(A) = \mathcal{K}(A)/\equiv_A \cong \mathcal{L}_2$*.*

Obviously, if A is Abelian, then A is nilpotent and solvable and has $\mathrm{Spec}(A) = \emptyset$*. Moreover, by [1, Proposition* 1.3*], if A is solvable or nilpotent, then* $\mathrm{Spec}(A) = \emptyset$*. Thus, if A is solvable or nilpotent, in particular if A is Abelian, then* $\mathcal{L}(A) \cong \mathcal{L}_1$*. For instance, according to [37], any Abelian group is an Abelian algebra, hence its reticulation is trivial.*

If A is simple, that is $\mathrm{Con}(A) = \{\Delta_A, \nabla_A\} \subseteq \mathcal{K}(A) \subseteq \mathrm{Con}(A)$*, so that* $\mathcal{K}(A) = \mathrm{Con}(A) = \{\Delta_A, \nabla_A\}$*, thus* $\mathcal{L}(A) = \{\mathbf{0}, \mathbf{1}\}$*, so we are situated in one of the following two cases: either A is Abelian, so that* $\mathcal{L}(A) \cong \mathcal{L}_1$*, or the commutator of A equals the intersection, so that* $\mathrm{Spec}(A) = \{\Delta_A\}$ *and thus* $\mathcal{L}(A) \cong \mathcal{L}_2$*.*

Proposition 15 *If the commutator of A equals the intersection, in particular if* $\mathcal{C}$ *is congruence–distributive, then* $\mathcal{K}(A)$ *is a bounded sublattice of the bounded distributive lattice* $\mathrm{Con}(A)$ *and* $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$ *is a lattice isomorphism, thus we may take* $\mathcal{L}(A) = \mathcal{K}(A)$*.*

Proof: Assume that $[\cdot, \cdot]_A = \cap$. $\Delta_A \in \mathrm{PCon}(A) \subseteq \mathcal{K}(A)$. $\mathcal{K}(A)$ is closed w.r.t. the join, and we are under the assumptions that $\nabla_A \in \mathcal{K}(A)$ and $\mathcal{K}(A)$ is closed w.r.t. the commutator, so w.r.t. the intersection. Hence

$\mathcal{K}(A)$ is a bounded sublattice of $\mathrm{Con}(A)$. By Lemma 8, $\equiv_A = \Delta_{\mathcal{K}(A)}$, thus $\mathcal{L}(A) = \mathcal{K}(A)/\Delta_{\mathcal{K}(A)} \cong \mathcal{K}(A)$ and the canonical surjection $\lambda_A : \mathcal{K}(A) \to \mathcal{L}(A)$ is a lattice isomorphism. □

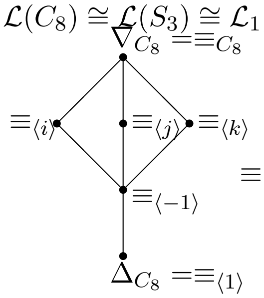
Remark 5 *If* $\mathrm{Con}(A) = \mathcal{K}(A)$*, in particular if* A *is finite, then* $\mathcal{L}(A) = \mathrm{Con}(A)/\equiv_A$*, so, if, furthermore, the commutator of* A *equals the intersection, in particular if* $\mathcal{C}$ *is congruence–distributive, then* $\mathcal{L}(A) \cong \mathrm{Con}(A)$ *by Proposition 15, thus we may take* $\mathcal{L}(A) = \mathrm{Con}(A)$*.*

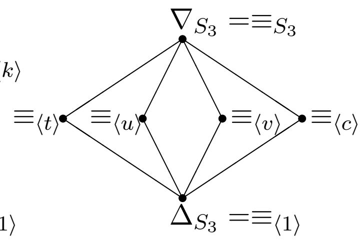
As a fact that may be interesting by its symmetry, if A *is finite and its commutator equals the intersection, so that* $\mathrm{Con}(A)$ *is a finite distributive lattice, then* $\mathcal{L}(\mathrm{Con}(A)) = \mathrm{Con}(\mathrm{Con}(A)) = \mathrm{Con}(\mathcal{L}(A))$*. It might also be interesting to find weaker conditions on* A *under which* $\mathcal{L}(\mathrm{Con}(A)) \cong \mathrm{Con}(\mathcal{L}(A))$*.*

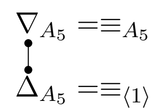
Remark 6 *By Proposition 15, if* A *is a residuated lattice, then* $\mathcal{L}(A) = \mathcal{K}(A)$*. If we denote by* $\mathrm{Filt}(A)$ *the set of the filters of* A *and by* $\mathrm{PFilt}(A)$ *the set of the principal filters of* A*, then, since* $\mathrm{Con}(A) \cong \mathrm{Filt}(A)$ *and the finitely generated filters of* A *are principal filters [21, 26], it follows that* $\mathcal{L}(A) = \mathcal{K}(A) \cong \mathrm{PFilt}(A)$*, which is the dual of the reticulation of a residuated lattice obtained in [39, 40, 41], where the reticulation has the prime spectrum of filters homeomorphic to the prime spectrum of filters, thus to that of congruences of* A *by the above, so this duality to the construction of* $\mathcal{L}(A)$ *from Section 4 was to be expected.*

Remark 7 *If* A *is a commutative unitary ring and* $\mathrm{Id}(A)$ *is its lattice of ideals, then it is well known that* $\mathrm{Id}(A) \cong \mathrm{Con}(A)$*. If, for all* $I \in \mathrm{Id}(A)$*, we denote by* $\sqrt{I}$ *the intersection of the prime filters of* A *which include* I*, then [8, Lemma, p. 1861] shows that, for any* $J \in \mathrm{Id}(A)$*, there exists a finitely generated ideal* K *of* A *such that* $\sqrt{J} = \sqrt{K}$*. From this, it immediately follows that the lattice* $\mathcal{L}(A)$ *is isomorphic to the reticulation of* A *constructed in [8].*

Example 2 *For any group* $(G, \cdot)$*, any* $x \in G$ *and any normal subgroup* H *of* G*, let us denote by* $\langle x \rangle$ *the subgroup of* G *generated by* x *and by* $\equiv_H$ *the congruence of* G *associated to* H*. As shown by the following commutators calculations, the cuaternions group,* $C_8 = \{1, -1, i, -i, j, -j, k, -k\}$*, is a solvable algebra which is not Abelian, while the group* $S_3 = \{1, t, u, v, c, d\}$ *of the permutations of the set* $\overline{1,3}$*, where* $1 = id_{\overline{1,3}}$*,* $t = (1\,2)$*,* $u = (1\,3)$*,* $v = (2\,3)$*,* $c = (1\,2\,3)$ *and* $d = c \circ c$*, has* $\mathrm{Spec}(S_3) = \emptyset$*, without being solvable or nilpotent. The following are the subgroups of* C_8*, respectively* S_3*, all of* *which are normal, and the proper ones are cyclic, thus Abelian:* $\langle 1\rangle$, $\langle -1\rangle$, $\langle i\rangle$, $\langle j\rangle$, $\langle k\rangle$ *and* C_8, *respectively* $\langle 1\rangle$, $\langle t\rangle$, $\langle u\rangle$, $\langle v\rangle$, $\langle c\rangle$ *and* S_3, *so* C_8 *and* S_3 *have the following congruence lattices and commutators, which suffice to conclude that* $\mathrm{Spec}(C_8) = \mathrm{Spec}(S_3) = \emptyset$, *since we are in a congruence–modular variety, and thus* $\mathcal{L}(C_8) \cong \mathcal{L}(S_3) \cong \mathcal{L}_1$, *by Remark 4:*







θ	$[\theta,\theta]_{C_8}$
Δ_{C_8}	Δ_{C_8}
$\equiv_{\langle -1\rangle}$	Δ_{C_8}
$\equiv_{\langle i\rangle}$	$\equiv_{\langle -1\rangle}$
$\equiv_{\langle j\rangle}$	$\equiv_{\langle -1\rangle}$
$\equiv_{\langle k\rangle}$	$\equiv_{\langle -1\rangle}$
∇_{C_8}	$\equiv_{\langle -1\rangle}$

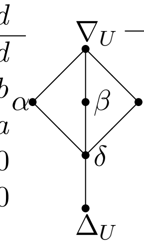
θ	$[\theta,\theta]_{S_3}$
Δ_{S_3}	Δ_{S_3}
$\equiv_{\langle t\rangle}$	Δ_{S_3}
$\equiv_{\langle u\rangle}$	Δ_{S_3}
$\equiv_{\langle v\rangle}$	Δ_{S_3}
$\equiv_{\langle c\rangle}$	Δ_{S_3}
∇_{S_3}	∇_{S_3}

θ	$[\theta,\theta]_{A_5}$
Δ_{A_5}	Δ_{A_5}
∇_{A_5}	∇_{A_5}

Here is an example of a group with non–empty prime spectrum, due to Erhard Aichinger and Bernhard Ganter: the group A_5 *of the even permutations of a* 5*–element set is the smallest non–Abelian simple group, thus it has the congruence lattice and the commutators above, where we have denoted by* 1 *the identity map of the* 5*–element set, so* $\mathrm{Spec}(A_5) = \{\Delta_{A_5}\}$, *thus* $\mathcal{L}(A_5) \cong \mathcal{L}_2$ *by Remark 4.*

Example 3 *This is the algebra from [2, Example 6.3] and [3, Example 4.2]:* $U = (\{0, a, b, c, d\}, +)$, *with* $+$ *defined by the following table, which has the congruence lattice represented below, where* $U/\alpha = \{\{0, a\}, \{b, c, d\}\}$, $U/\beta = \{\{0, b\}, \{a, c, d\}\}$, $U/\gamma = \{\{0, c, d\}, \{a, b\}\}$ *and* $U/\delta = \{\{0\}, \{a\}, \{b\}, \{c, d\}\}$*:*

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	0	c	b	b
b	b	c	0	a	a
c	c	b	a	0	0
d	d	b	a	0	0

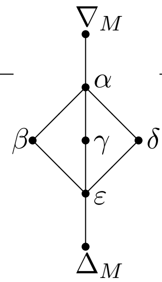


$[\cdot,\cdot]_U$	Δ_U	α	β	γ	δ	∇_U
Δ_U	Δ_U	Δ_U	Δ_U	Δ_U	Δ_U	Δ_U
α	Δ_U	δ	δ	δ	δ	α
β	Δ_U	δ	δ	δ	δ	β
γ	Δ_U	δ	δ	γ	δ	γ
δ	Δ_U	δ	δ	δ	Δ_U	δ
∇_U	Δ_U	α	β	γ	δ	∇_U

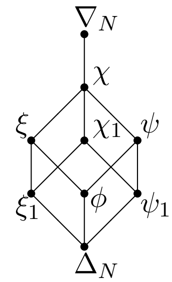
U *is not Abelian, nor is it solvable or nilpotent, as shown by the table of* $[\cdot,\cdot]_U$ *above, but* $\mathrm{Spec}(U) = \emptyset$, *thus* $\mathcal{L}(U) \cong \mathcal{L}_1$ *by Remark 4.*

Example 4 *Let $M=(\{a,b,x,y,z\},+)$ and $N=(\{a,b,c,x,y\},+)$, with $+$ defined by the following tables. Then $\mathrm{Con}(M)$ and $\mathrm{Con}(N)$ have the Hasse diagrams below, where: $M/\alpha=\{\{a,b\},\{x,y,z\}\}$, $M/\beta=\{\{a,b\},\{x,y\},\{z\}\}$, $M/\gamma=\{\{a,b\},\{x,z\},\{y\}\}$, $M/\delta=\{\{a,b\},\{x\},\{y,z\}\}$ and $M/\varepsilon=\{\{a,b\},\{x\},\{y\},\{z\}\}$, $N/\chi=\{\{a,b,c\},\{x,y\}\}$, $N/\chi_1=\{\{a,b,c\},\{x\},\{y\}\}$, $N/\xi=\{\{a,b\},\{c\},\{x,y\}\}$, $N/\xi_1=\{\{a,b\},\{c\},\{x\},\{y\}\}$, $N/\psi=\{\{a\},\{b,c\},\{x,y\}\}$, $N/\psi_1=\{\{a\},\{b,c\},\{x\},\{y\}\}$ and $N/\phi=\{\{a\},\{b\},\{c\},\{x,y\}\}$.*

$+$	a	b	x	y	z
a	a	b	a	a	a
b	b	b	b	b	b
x	x	x	x	x	x
y	y	y	y	y	y
z	z	z	z	z	z



$+$	a	b	c	x	y
a	a	b	c	a	a
b	b	b	c	b	b
c	c	c	c	c	c
x	x	x	x	x	x
y	y	y	y	y	y



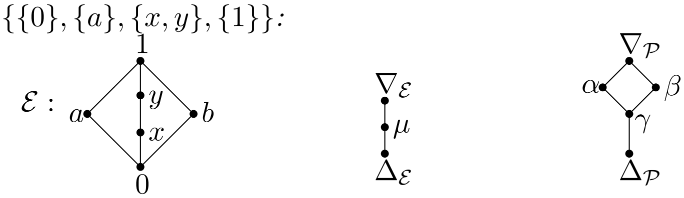
Although M is congruence–modular and N is congruence–distributive, neither $\mathcal{HSP}(M)$, nor $\mathcal{HSP}(N)$ is congruence–modular, because $S=(\{a,b\},+)\cong(\mathcal{L}_2,\max)\cong(\mathbb{Z}_2,\cdot)$ is a subalgebra of both M and N, and it can be easily checked that S^2 is not congruence–modular. Thus neither $\mathcal{HSP}(M)$, nor $\mathcal{HSP}(N)$ is semi–degenerate, which is also obvious from the fact that $(\{a\},+)$ is a subalgebra of both M and N.

We have: $[\theta,\zeta]_M=\varepsilon$ for all $\theta,\zeta\in[\varepsilon)$ and, of course, $[\Delta_M,\theta]_M=[\theta,\Delta_M]_M=\Delta_M$ for all $\theta\in\mathrm{Con}(M)$, hence $\mathrm{Spec}(M)=\{\Delta_M\}$ and thus $\mathcal{L}(M)\cong\mathcal{L}_2$, while $[\cdot,\cdot]_N$ is given by the following table, thus $\mathrm{Spec}(N)=\{\psi,\xi\}$, so ρ_N is defined as follows and hence $\mathcal{L}(N)=\{\mathbf{0},\widehat{\xi},\widehat{\psi},\mathbf{1}\}\cong\mathcal{L}_2^2$:

$[\cdot,\cdot]_N$	Δ_N	ψ	ψ_1	ϕ	ξ	ξ_1	χ	χ_1	∇_N	$\rho_N(\cdot)$
Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	ϕ
ψ	Δ_N	ψ_1	ψ_1	Δ_N	Δ_N	Δ_N	ψ_1	ψ_1	ψ_1	ψ
ψ_1	Δ_N	ψ_1	ψ_1	Δ_N	Δ_N	Δ_N	ψ_1	ψ_1	ψ_1	ψ
ϕ	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	Δ_N	ϕ
ξ	Δ_N	Δ_N	Δ_N	Δ_N	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ
ξ_1	Δ_N	Δ_N	Δ_N	Δ_N	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ
χ	Δ_N	ψ_1	ψ_1	Δ_N	ξ_1	ξ_1	χ_1	χ_1	χ_1	∇_N
χ_1	Δ_N	ψ_1	ψ_1	Δ_N	ξ_1	ξ_1	χ_1	χ_1	χ_1	∇_N
∇_N	Δ_N	ψ_1	ψ_1	Δ_N	ξ_1	ξ_1	χ_1	χ_1	χ_1	∇_N

Example 5 *Here are some finite examples in the congruence–distributive variety of lattices, thus in which the reticulations are isomorphic to the congruence lattices. Regarding the preservation properties fulfilled by the reticulation,*

these examples show that there is no embedding relation between the reticulation of an algebra and those of its subalgebras: if $\mathcal{E}$ is the following bounded lattice, then, for instance, $\{0, x, y, 1\} = \mathcal{L}_4 = \mathcal{L}_2 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2$, $\mathcal{D} = \{0, a, x, b, 1\}$ and $\mathcal{P} = \{0, a, x, y, 1\}$ are bounded sublattices of $\mathcal{E}$. We have: $\mathcal{L}(\mathcal{E}) \cong \mathrm{Con}(\mathcal{E}) = \{\Delta_{\mathcal{E}}, \mu, \nabla_{\mathcal{E}}\} \cong \mathcal{L}_3$, where $\mathcal{E}/\mu = \{\{0\}, \{a\}, \{x, y\}, \{b\}, \{1\}\}$, $\mathcal{L}(\mathcal{L}_4) \cong \mathrm{Con}(\mathcal{L}_4) = \mathrm{Con}(\mathcal{L}_2 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2) \cong \mathrm{Con}(\mathcal{L}_2)^3 \cong \mathcal{L}_2^3$, $\mathcal{L}(\mathcal{D}) \cong \mathrm{Con}(\mathcal{D}) = \{\Delta_{\mathcal{D}}, \nabla_{\mathcal{D}}\} \cong \mathcal{L}_2$ and $\mathcal{L}(\mathcal{P}) \cong \mathrm{Con}(\mathcal{P}) = \{\Delta_{\mathcal{P}}, \alpha, \beta, \gamma, \nabla_{\mathcal{P}}\} \cong \mathcal{L}_2 \oplus \mathcal{L}_2^2$, where $\mathcal{P}/\alpha = \{\{0, x, y\}, \{a, 1\}\}$, $\mathcal{P}/\beta = \{\{0, a\}, \{x, y, 1\}\}$ and $\mathcal{P}/\gamma = \{\{0\}, \{a\}, \{x, y\}, \{1\}\}$:



Remark 8 *By [22, Lemma 3.3], in any variety, arbitrary intersections commute with arbitrary direct products of congruences. If $\mathcal{C}$ is congruence–modular and M is an algebra from $\mathcal{C}$ such that $A \times M$ has no skew congruences, then $\mathrm{Spec}(A \times M) = \{\phi \times \nabla_M \mid \phi \in \mathrm{Spec}(A)\} \cup \{\nabla_A \times \psi \mid \psi \in \mathrm{Spec}(M)\}$. This follows from Proposition 1 in the same way as in the congruence–distributive case, treated in [22, Proposition 3.5,(ii)].*

Now let us see that the reticulation preserves finite direct products without skew congruences:

Theorem 4 *Let M be an algebra from $\mathcal{C}$ such that the direct product $A \times M$ has no skew congruences. Then:*

(i) for all $\emptyset \neq X \subseteq A^2$ and all $\emptyset \neq Y \subseteq M^2$, $Cg_{A \times M}(X \times Y) = Cg_A(X) \times Cg_M(Y)$, and the map $(\alpha, \mu) \mapsto \alpha \times \mu$ is a lattice isomorphism from $\mathrm{Con}(A) \times \mathrm{Con}(M)$ to $\mathrm{Con}(A \times M)$;

(ii) $\mathrm{PCon}(A \times M) = \{\alpha \times \mu \mid \alpha \in \mathrm{PCon}(A), \mu \in \mathrm{PCon}(M)\}$ and
$\mathcal{K}(A \times M) = \{\alpha \times \mu \mid \alpha \in \mathcal{K}(A), \mu \in \mathcal{K}(M)\}$.

If $\mathcal{C}$ is congruence–modular and $\nabla_M \in \mathcal{K}(M)$, then:

- *for all $\alpha \in \mathrm{Con}(A)$ and all $\mu \in \mathrm{Con}(M)$,*
  $\rho_{A \times M}(\alpha \times \mu) = \rho_A(\alpha) \times \rho_M(\mu)$;
- *$\equiv_{A \times M} = \equiv_A \times \equiv_M$ and $\mathcal{L}(A \times M) \cong \mathcal{L}(A) \times \mathcal{L}(M)$.*

Proof: $A \times M$ has no skew congruences, that is $\mathrm{Con}(A \times M) = \{\alpha \times \mu \mid \alpha \in \mathrm{Con}(A), \mu \in \mathrm{Con}(M)\}$.

(i) By Remark 8, $Cg_{A\times M}(X \times Y) = \bigcap\{\theta \in \mathrm{Con}(A \times M) \mid X \times Y \subseteq \theta\} = \bigcap\{\alpha \times \mu \mid \alpha \in \mathrm{Con}(A), \mu \in \mathrm{Con}(M), X \times Y \subseteq \alpha \times \mu\} = \bigcap\{\alpha \times \mu \mid \alpha \in \mathrm{Con}(A), \mu \in \mathrm{Con}(M), X \subseteq \alpha, Y \subseteq \mu\} = (\bigcap\{\alpha \in \mathrm{Con}(A) \mid X \subseteq \alpha\}) \times (\bigcap\{\mu \in \mathrm{Con}(M) \mid Y \subseteq \mu\}) = Cg_A(X) \times Cg_M(Y)$.

This also shows that the map $(\alpha, \mu) \mapsto \alpha \times \mu$ is a lattice isomorphism from $\mathrm{Con}(A) \times \mathrm{Con}(M)$ to $\mathrm{Con}(A \times M)$, because it is clearly injective, it is surjective by the above, it preserves the intersection by Remark 8 and, for all $\alpha, \beta \in \mathrm{Con}(A)$ and all $\mu, \nu \in \mathrm{Con}(M)$, $(\alpha \times \mu) \vee (\beta \times \nu) = Cg_{A\times M}((\alpha \times \mu) \cup(\beta \times \nu)) \subseteq Cg_{A\times M}((\alpha \cup \beta) \times (\mu \cup \nu)) = Cg_A(\alpha \cup \beta) \times Cg_M(\mu \cup \nu) = (\alpha \vee \beta) \times (\mu \vee \nu)$, since, clearly, $(\alpha \times \mu) \cup (\beta \times \nu) \subseteq (\alpha \cup \beta) \times (\mu \cup \nu)$, but, also, $(\alpha \times \mu) \vee (\beta \times \nu) \in \mathrm{Con}(A \times M)$, thus $(\alpha \times \mu) \cup (\beta \times \nu) \subseteq (\alpha \times \mu) \vee (\beta \times \nu) = \gamma \times \sigma$ for some $\gamma \in \mathrm{Con}(A)$ and $\sigma \in \mathrm{Con}(M)$, so $\alpha \times \mu \subseteq \gamma \times \sigma$ and $\beta \times \nu \subseteq \gamma \times \sigma$, hence $\alpha \subseteq \gamma$, $\beta \subseteq \gamma$, $\mu \subseteq \sigma$ and $\nu \subseteq \sigma$, so $\alpha \vee \beta \subseteq \gamma$ and $\mu \vee \nu \subseteq \sigma$, hence $(\alpha \vee \beta) \times (\mu \vee \nu) \subseteq \gamma \times \sigma = (\alpha \times \mu) \vee (\beta \times \nu) \subseteq (\alpha \vee \beta) \times (\mu \vee \nu)$, therefore $(\alpha \vee \beta) \times (\mu \vee \nu) = (\alpha \times \mu) \vee (\beta \times \nu)$.

(ii) By (i), for all $a, b \in A$ and all $u, v \in M$, $Cg_{A\times M}((a, u), (b, v)) = Cg_A(a, b) \times Cg_M(u, v)$, hence the expression of $\mathrm{PCon}(A \times M)$ in the enunciation. From this and the second statement in (i), we obtain: $\mathcal{K}(A \times M)$ $= \{Cg_{A\times M}(\{(a_1, u_1), \ldots, (a_n, u_n)\}) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\}$ $= \{\bigvee_{i=1}^n Cg_{A\times M}(a_i, u_i) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\}$ $= \{\bigvee_{i=1}^n (Cg_A(a_i) \times Cg_M(u_i)) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\}$ $= \{(\bigvee_{i=1}^n Cg_A(a_i)) \times (\bigvee_{i=1}^n Cg_M(u_i)) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\}$ $= \{Cg_A(a_1, \ldots, a_n) \times Cg_M(u_1, \ldots, u_n) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\}$ $= \{\alpha \times \mu \mid \alpha \in \mathcal{K}(A), \mu \in \mathcal{K}(M)\}$.

Now assume that $\mathcal{C}$ is congruence–modular and $\nabla_M \in \mathcal{K}(M)$. Then, by Remark 8, for any $\alpha \in \mathrm{Con}(A)$ and any $\mu \in \mathrm{Con}(M)$, $\rho_{A\times M}(\alpha \times \mu) = \bigcap\{\chi \in \mathrm{Spec}(A \times M) \mid \alpha \times \mu \subseteq \chi\} = \bigcap\{\phi \times \nabla_M \mid \phi \in \mathrm{Spec}(A), \alpha \times \mu \subseteq \phi \times \nabla_M\} \cap \bigcap\{\nabla_A \times \psi \mid \psi \in \mathrm{Spec}(M), \alpha \times \mu \subseteq \nabla_A \times \psi\} = \bigcap\{\phi \times \nabla_M \mid \phi \in \mathrm{Spec}(A), \alpha \subseteq \phi\} \cap \bigcap\{\nabla_A \times \psi \mid \psi \in \mathrm{Spec}(M), \mu \subseteq \psi\} = (\bigcap\{\phi \mid \phi \in \mathrm{Spec}(A), \alpha \subseteq \phi\} \times \nabla_M) \cap (\nabla_A \times \bigcap\{\psi \mid \psi \in \mathrm{Spec}(M), \mu \subseteq \psi\}) = (\rho_A(\alpha) \times \nabla_M) \cap (\nabla_A \times \rho_M(\mu)) = (\rho_A(\alpha) \cap \nabla_A) \times (\nabla_M \cap \rho_M(\mu)) = \rho_A(\alpha) \times \rho_M(\mu)$. Hence, for all $\theta, \zeta \in \mathrm{Con}(A \times M)$, we have: $\theta = \alpha \times \mu$ and $\zeta = \beta \times \nu$ for some $\alpha, \beta \in \mathrm{Con}(A)$ and $\mu, \nu \in \mathrm{Con}(M)$, and thus: $\theta \equiv_{A\times M} \zeta$ iff $\rho_{A\times M}(\theta) = \rho_{A\times M}(\zeta)$ iff $\rho_{A\times M}(\alpha \times \mu) = \rho_{A\times M}(\beta \times \nu)$ iff $\rho_A(\alpha) \times \rho_M(\mu) = \rho_A(\beta) \times \rho_M(\nu)$ iff $\rho_A(\alpha) = \rho_A(\beta)$ and $\rho_M(\mu) = \rho_M(\nu)$ iff $\alpha \equiv_A \beta$ and $\mu \equiv_M \nu$.

Now let $\varphi : \mathcal{L}(A) \times \mathcal{L}(M) \to \mathcal{L}(A \times M)$, for all $\alpha \in \mathcal{K}(A)$ and all

$\mu \in \mathcal{K}(M)$, $\varphi(\widehat{\alpha}, \widehat{\mu}) = \widehat{\alpha \times \mu}$. By (ii), φ is well defined and surjective and fulfills, for all $\alpha, \beta \in \mathcal{K}(A)$ and all $\mu, \nu \in \mathcal{K}(M)$: $\varphi((\widehat{\alpha}, \widehat{\mu}) \vee (\widehat{\beta}, \widehat{\nu})) = \varphi(\widehat{\alpha} \vee \widehat{\beta}, \widehat{\mu} \vee \widehat{\nu}) = \varphi(\widehat{\alpha \vee \beta}, \widehat{\mu \vee \nu}) = ((\alpha \vee \beta) \times (\mu \vee \nu))^\wedge = ((\alpha \times \mu) \vee (\beta \times \nu))^\wedge = \widehat{(\alpha \times \mu)} \vee \widehat{(\beta \times \nu)} = \varphi(\widehat{\alpha}, \widehat{\mu}) \vee \varphi(\widehat{\beta}, \widehat{\nu})$ and, similarly, $\varphi((\widehat{\alpha}, \widehat{\mu}) \wedge (\widehat{\beta}, \widehat{\nu})) = \varphi(\widehat{\alpha}, \widehat{\mu}) \wedge \varphi(\widehat{\beta}, \widehat{\nu})$. By the form of $\equiv_{A \times M}$ above, φ is injective. Hence φ is a lattice isomorphism. □

Example 6 *Let $\mathcal{V}$ be the variety generated by the variety of lattices and that of groups. Then, according to [14, Theorem 1, Lemma 1, Proposition 3] and [33], $\mathcal{V}$ is congruence–modular and any algebra M from $\mathcal{V}$ is of the form $M = (L, \vee, \wedge) \times (G, \cdot, \star)$, where $(L, \vee, \wedge)$ is a lattice, $(G, \cdot)$ is a group and $x \star y = x^{-1} \cdot y$ for all $x, y \in G$, and the direct product above has no skew congruences, thus, by Theorem 4, $\mathrm{Con}(M) \cong \mathrm{Con}(L) \times \mathrm{Con}(G)$ and $\mathcal{L}(M) \cong \mathcal{L}(L) \times \mathcal{L}(G)$, since each congruence of the group G also preserves the operation $\star$. Thus, for instance, if we consider the lattice $\mathcal{P}$ from Example 5 and the group $(S_3, \circ)$ from Example 2, and we denote $\sigma \star \tau = \sigma^{-1} \circ \tau$ for all $\sigma, \tau \in S_3$, and $M = (\mathcal{P}, \vee, \wedge) \times (S_3, \circ, \star)$, then M is a finite algebra from $\mathcal{V}$ which is not congruence–distributive, because $\mathrm{Con}(M) \cong \mathrm{Con}(\mathcal{P}) \times \mathrm{Con}(S_3)$ and $\mathrm{Con}(S_3)$ is not distributive, and $\mathcal{L}(M) \cong \mathcal{L}(\mathcal{P}) \times \mathcal{L}(S_3) \cong \mathcal{L}(\mathcal{P}) \times \mathcal{L}_1 \cong \mathcal{L}(\mathcal{P}) \cong \mathrm{Con}(\mathcal{P}) \cong \mathcal{L}_2 \oplus \mathcal{L}_2^2$.*

6 Boolean Congruences versus the Reticulation

Throughout this section, we shall assume that $[\cdot, \cdot]_A$ is commutative and distributive w.r.t. arbitrary joins and $\nabla_A \in \mathcal{K}(A)$. For any bounded lattice L, we shall denote by $\mathcal{B}(L)$ the set of the complemented elements of L. If L is distributive, then $\mathcal{B}(L)$ is the Boolean center of L. Although $\mathrm{Con}(A)$ is not necessarily distributive, we shall call $\mathcal{B}(\mathrm{Con}(A))$ the *Boolean center* of $\mathrm{Con}(A)$.

Lemma 15 *For all $n \in \mathbb{N}^*$ and all $\alpha, \beta \in \mathrm{Con}(A)$: $[\alpha, \beta]_A^{n+1} = [[\alpha, \beta]_A, [\alpha, \beta]_A]_A^n$; if the commutator of A is associative, then $[\alpha, \beta]_A^{n+1} = [[\alpha, \alpha]_A^n, [\beta, \beta]_A^n]_A$.*

Proof: By induction on n. □

Lemma 16 *For all $n, k \in \mathbb{N}^*$ and all $\alpha, \beta, \phi, \psi, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathrm{Con}(A)$:*

(i) if $\alpha \subseteq \beta$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_A^n \subseteq [\beta, \psi]_A^n$;

(ii) *if* $k \leq n$*, then* $[\alpha, \beta]_A^n \subseteq [\alpha, \beta]_A^k$*;*

(iii) *if* $k \geq 2$ *and* $n \geq 2$*, then* $[\alpha, \beta]_A^{k \cdot n} \subseteq [[\alpha, \beta]_A^k, [\alpha, \beta]_A^k]_A^n$*;*

(iv) $[\alpha \vee \beta, \alpha \vee \beta]_A^n \subseteq \alpha \vee [\beta, \beta]_A^n$*;*

(v) $[\alpha \vee \beta, \alpha \vee \beta]_A^{n \cdot k} \subseteq [\alpha, \alpha]_A^k \vee [\beta, \beta]_A^n$ *and* $[\alpha \vee \beta, \alpha \vee \beta]_A^{n^2} \subseteq [\alpha, \alpha]_A^n \vee [\beta, \beta]_A^n$*;*

(vi) $[\alpha_1 \vee \ldots \vee \alpha_k, \alpha_1 \vee \ldots \vee \alpha_k]_A^{n^k} \subseteq [\alpha_1, \alpha_1]_A^n \vee \ldots \vee [\alpha_k, \alpha_k]_A^n$.

Proof: (i) By induction on n.
(ii) For all $p \in \mathbb{N}^*$, $[\alpha, \beta]_A^{p+1} = [[\alpha, \beta]_A^p, [\alpha, \beta]_A^p]_A \subseteq [\alpha, \beta]_A^p$, hence the inclusion in the enunciation.
(iii) Assume that $n \geq 2$. We apply induction on k, (ii) and Lemma 15. For $k = 2$, we have: $[[\alpha, \beta]_A^2, [\alpha, \beta]_A^2]_A^n = [[[\alpha, \beta]_A, [\alpha, \beta]_A]_A, [[\alpha, \beta]_A, [\alpha, \beta]_A]_A]_A^n = [\alpha, \beta]_A^{n+2} \supseteq [\alpha, \beta]_A^{2n}$. Now take a $k \geq 2$ that fulfills the inclusion in the enunciation for all $\alpha, \beta \in \mathrm{Con}(A)$. Then $[[\alpha, \beta]_A^{k+1}, [\alpha, \beta]_A^{k+1}]_A^n = [[[\alpha, \beta]_A, [\alpha, \beta]_A]_A^k, [[\alpha, \beta]_A, [\alpha, \beta]_A]_A^k]_A^n \supseteq [[\alpha, \beta]_A, [\alpha, \beta]_A]_A^{k \cdot n} = [\alpha, \beta]_A^{k \cdot n + 1} \supseteq [\alpha, \beta]_A^{k \cdot n + n} = [\alpha, \beta]_A^{(k+1) \cdot n}$.
(iv) We apply induction on n. For $n = 1$ and all $\alpha, \beta \in \mathrm{Con}(A)$, we have $[\alpha \vee \beta, \alpha \vee \beta]_A = [\alpha, \alpha]_A \vee [\alpha, \beta]_A \vee [\beta, \alpha]_A \vee [\beta, \beta]_A \subseteq \alpha \vee [\beta, \beta]_A$. Now let $n \in \mathbb{N}^*$ such that $[\alpha \vee \beta, \alpha \vee \beta]_A^n \subseteq \alpha \vee [\beta, \beta]_A^n$ for all $\alpha, \beta \in \mathrm{Con}(A)$. Then, by the induction hypothesis and the case $n = 1$, we have, for all $\alpha, \beta \in \mathrm{Con}(A)$: $[\alpha \vee \beta, \alpha \vee \beta]_A^{n+1} = [[\alpha \vee \beta, \alpha \vee \beta]_A^n, [\alpha \vee \beta, \alpha \vee \beta]_A^n]_A \subseteq [\alpha \vee [\beta, \beta]_A^n, \alpha \vee [\beta, \beta]_A^n]_A \subseteq \alpha \vee [[\beta, \beta]_A^n, [\beta, \beta]_A^n]_A = \alpha \vee [\beta, \beta]_A^{n+1}$.
(v) We apply (iv) and (iii). For $n = 1$, $[\alpha, \alpha]_A^k \vee [\beta, \beta]_A \supseteq [\alpha \vee \beta, \alpha \vee \beta]_A^k$. For $k = 1$, $[\alpha, \alpha]_A \vee [\beta, \beta]_A^n \supseteq [\alpha \vee \beta, \alpha \vee \beta]_A^n$. For $k \geq 2$ and $n \geq 2$, $[\alpha, \alpha]_A^k \vee [\beta, \beta]_A^n \supseteq [[\alpha, \alpha]_A^k \vee \beta, [\alpha, \alpha]_A^k \vee \beta]_A^n \supseteq [[\alpha \vee \beta, \alpha \vee \beta]_A^k, [\alpha \vee \beta, \alpha \vee \beta]_A^k]_A^n \supseteq [\alpha \vee \beta, \alpha \vee \beta]_A^{k \cdot n}$. For $k = n$, we obtain the second equality.
(vi) We apply induction on k. The statement is trivial for $k = 1$. Let $k \in \mathbb{N}^*$ that fulfills the equality in the enunciation for any congruences of A, and let $\alpha_1, \ldots, \alpha_k, \alpha_{k+1} \in \mathrm{Con}(A)$. By (v), it follows that $[\alpha_1 \vee \ldots \vee \alpha_k \vee \alpha_{k+1}, \alpha_1 \vee \ldots \vee \alpha_k \vee \alpha_{k+1}]_A^{n^{k+1}} = [(\alpha_1 \vee \ldots \vee \alpha_k) \vee \alpha_{k+1}, (\alpha_1 \vee \ldots \vee \alpha_k) \vee \alpha_{k+1}]_A^{n^k \cdot n} \subseteq [\alpha_1 \vee \ldots \vee \alpha_k, \alpha_1 \vee \ldots \vee \alpha_k]_A^{n^k} \vee [\alpha_{k+1}, \alpha_{k+1}]_A^n \subseteq [\alpha_1, \alpha_1]_A^n \vee \ldots \vee [\alpha_k, \alpha_k]_A^n \vee [\alpha_{k+1}, \alpha_{k+1}]_A^n$. □

For the following, recall, also, the equivalences in Proposition 3.

For all $\theta, \zeta \in \mathrm{Con}(A)$, we shall denote by $\theta \rightarrow \zeta = \bigvee \{\alpha \in \mathrm{Con}(A) \mid [\theta, \alpha]_A \subseteq \zeta\}$ and by $\theta^\perp = \theta \rightarrow \Delta_A = \bigvee \{\alpha \in \mathrm{Con}(A) \mid [\theta, \alpha]_A = \Delta_A\}$.

Lemma 17 *For all $\alpha, \beta, \gamma \in \mathrm{Con}(A)$:*

(i) $[\alpha, \beta]_A \subseteq \gamma$ *iff* $\alpha \subseteq \beta \rightarrow \gamma$*;*

(ii) *if* $[\alpha, \nabla_A]_A = \alpha$*, then:* $\alpha \rightarrow \beta = \nabla_A$ *iff* $\alpha \subseteq \beta$*.*

Proof: (i) "⇒:" $\beta \rightarrow \gamma = \bigvee \{\theta \in \mathrm{Con}(A) \mid [\beta, \theta]_A \subseteq \gamma\}$. Since $[\beta, \alpha]_A = [\alpha, \beta]_A \subseteq \gamma$, it follows that $\alpha \subseteq \beta \rightarrow \gamma$.
"⇐:" We have $\alpha \subseteq \beta \rightarrow \gamma = \bigvee \{\theta \in \mathrm{Con}(A) \mid [\beta, \theta]_A \subseteq \gamma\}$, hence $[\beta, \alpha]_A = [\alpha, \beta]_A \subseteq [\beta, \beta \rightarrow \gamma]_A = [\beta, \bigvee \{\theta \in \mathrm{Con}(A) \mid [\beta, \theta]_A \subseteq \gamma\}]_A = \bigvee \{[\beta, \theta]_A \mid \theta \in \mathrm{Con}(A), [\beta, \theta]_A \subseteq \gamma\} \subseteq \gamma$.
(ii) By (i), $\alpha \rightarrow \beta = \nabla_A$ iff $\nabla_A \subseteq \alpha \rightarrow \beta$ iff $\alpha = [\nabla_A, \alpha]_A \subseteq \beta$. □

Remark 9 *For all $\theta, \zeta \in \mathrm{Con}(A)$, $\theta \rightarrow \zeta = \max\{\alpha \in \mathrm{Con}(A) \mid [\theta, \alpha]_A \subseteq \zeta\}$, because, if we denote by $M = \{\alpha \in \mathrm{Con}(A) \mid [\theta, \alpha]_A \subseteq \zeta\}$, then $[\theta, \theta \rightarrow \zeta]_A = [\theta, \bigvee_{\alpha \in M} \alpha]_A = \bigvee_{\alpha \in M} [\theta, \alpha]_A \subseteq \zeta$, hence $\theta \rightarrow \zeta \in M$.*

Note, also, from the above, that, if $[\cdot, \cdot]_A$ is associative and $[\alpha, \nabla_A]_A = \alpha$ for all $\alpha \in \mathrm{Con}(A)$, then $(\mathrm{Con}(A), \vee, \cap, [\cdot, \cdot]_A, \rightarrow, \Delta_A, \nabla_A)$ is a residuated lattice, which, of course, is a Gödel algebra if $\cap = [\cdot, \cdot]_A$, in particular if $\mathcal{C}$ is congruence–distributive.

Lemma 18 *If $[\theta, \nabla_A]_A = \theta$ for all $\theta \in \mathrm{Con}(A)$, then, for any $\alpha, \beta, \gamma \in \mathrm{Con}(A)$ and any $\sigma \in \mathcal{B}(\mathrm{Con}(A))$:*

(i) *if* $\alpha \vee \beta = \nabla_A$*, then* $[\alpha, \beta]_A = \alpha \cap \beta$*;*

(ii) *if* $\alpha \vee \beta = \alpha \vee \gamma = \nabla_A$*, then* $\alpha \vee [\beta, \gamma]_A = \alpha \vee (\beta \cap \gamma) = \nabla_A$*;*

(iii) *if* $\alpha \vee \beta = \nabla_A$*, then* $[\alpha, \alpha]_A^n \vee [\beta, \beta]_A^n = \nabla_A$ *for all* $n \in \mathbb{N}^*$*;*

(iv) $[\sigma, \alpha]_A = \sigma \cap \alpha$*.*

Proof: (i) Assume that $\alpha \vee \beta = \nabla_A$. Since $(\alpha \cap \beta) \rightarrow [\alpha, \beta]_A = \bigvee \{\theta \in \mathrm{Con}(A) \mid [\alpha \cap \beta, \theta]_A \subseteq [\alpha, \beta]_A\}$ and $[\alpha \cap \beta, \beta]_A \subseteq [\alpha, \beta]_A$ and $[\alpha, \alpha \cap \beta]_A \subseteq [\alpha, \beta]_A$, it follows that $\alpha \subseteq (\alpha \cap \beta) \rightarrow [\alpha, \beta]_A$ and $\beta \subseteq (\alpha \cap \beta) \rightarrow [\alpha, \beta]_A$, hence $\nabla_A = \alpha \vee \beta \subseteq (\alpha \cap \beta) \rightarrow [\alpha, \beta]_A$, therefore $(\alpha \cap \beta) \rightarrow [\alpha, \beta]_A = \nabla_A$, thus $\alpha \cap \beta \subseteq [\alpha, \beta]_A$ by Lemma 17, (ii). Since the converse inclusion always holds, it follows that $\alpha \cap \beta = [\alpha, \beta]_A$.

(ii) Assume that $\alpha\vee\beta=\alpha\vee\gamma=\nabla_A$, so that $\nabla_A=[\nabla_A,\nabla_A]_A=[\alpha\vee\beta,\alpha\vee\gamma]_A=[\alpha,\alpha]_A\vee[\beta,\alpha]_A\vee[\alpha,\gamma]_A\vee[\beta,\gamma]_A\subseteq\alpha\vee[\beta,\gamma]_A\subseteq\alpha\vee(\beta\cap\gamma)\subseteq\nabla_A$, hence $\alpha\vee[\beta,\gamma]_A=\alpha\vee(\beta\cap\gamma)=\nabla_A$.
(iii) We apply induction on n. Assume that $\alpha\vee\beta=\nabla_A$, so that, by (ii), $\alpha\vee[\beta,\beta]_A=\nabla_A$, thus $[\alpha,\alpha]_A\vee[\beta,\beta]_A=\nabla_A$, hence the implication holds in the case $n=1$. Now, if $n\in\mathbb{N}^*$ fulfills the implication in the enunciation for all $\alpha,\beta\in\mathrm{Con}(A)$, and assume that $\alpha\vee\beta=\nabla_A$, so that $[\alpha,\alpha]_A^n\vee[\beta,\beta]_A^n=\nabla_A$. Then, by the case $n=1$, it follows that $[\alpha,\alpha]_A^{n+1}\vee[\beta,\beta]_A^{n+1}=[[\alpha,\alpha]_A^n,[\alpha,\alpha]_A^n]_A\vee[[\beta,\beta]_A^n,[\beta,\beta]_A^n]_A=\nabla_A$.
(iv) $\sigma\in\mathcal{B}(\mathrm{Con}(A))$, so there exists a $\tau\in\mathrm{Con}(A)$ with $\sigma\vee\tau=\nabla_A$ and $\sigma\cap\tau=\Delta_A$. Thus the following hold: $\sigma\cap\alpha=[\nabla_A,\sigma\cap\alpha]_A=[\sigma\vee\tau,\sigma\cap\alpha]_A=[\sigma,\sigma\cap\alpha]_A\vee[\tau,\sigma\cap\alpha]_A\subseteq[\sigma,\alpha]_A\vee(\tau\cap\sigma\cap\alpha)=[\sigma,\alpha]_A\vee\Delta_A=[\sigma,\alpha]_A\subseteq\sigma\cap\alpha$, hence $[\sigma,\alpha]_A=\sigma\cap\alpha$. For (iv), we have followed the argument from [27, Lemma 4]. □

Remark 10 *By Lemma 18, (iv), if $[\theta,\nabla_A]_A=\theta$ for all $\theta\in\mathrm{Con}(A)$, then, in $\mathcal{B}(\mathrm{Con}(A))$, the commutator of A equals the intersection, in particular the intersection in $\mathcal{B}(\mathrm{Con}(A))$ is distributive with respect to the join.*

Lemma 19 *If f is surjective, then: $f(\mathrm{PCon}(A)\cap[\mathrm{Ker}(f)))\subseteq f(\{\alpha\vee\mathrm{Ker}(f)\mid\alpha\in\mathrm{PCon}(A)\})=\mathrm{PCon}(B)$ and $f(\mathcal{K}(A)\cap[\mathrm{Ker}(f)))\subseteq f(\{\alpha\vee\mathrm{Ker}(f)\mid\alpha\in\mathcal{K}(A)\})=\mathcal{K}(B)$; if $\mathcal{C}$ is congruence–modular and semi–degenerate, then $f(\mathcal{B}(\mathrm{Con}(A))\cap[\mathrm{Ker}(f)))\subseteq f(\{\alpha\vee\mathrm{Ker}(f)\mid\alpha\in\mathcal{B}(\mathrm{Con}(A))\})\subseteq\mathcal{B}(\mathrm{Con}(B))$.*

In particular, for all $\theta\in\mathrm{Con}(A)$: $\{\alpha/\theta\mid\alpha\in\mathrm{PCon}(A)\cap[\theta)\}\subseteq\{(\alpha\vee\theta)/\theta\mid\alpha\in\mathrm{PCon}(A)\}=\mathrm{PCon}(A/\theta)$ and $\{\alpha/\theta\mid\alpha\in\mathcal{K}(A)\cap[\theta)\}\subseteq\{(\alpha\vee\theta)/\theta\mid\alpha\in\mathcal{K}(A)\}=\mathcal{K}(A/\theta)$; if $\mathcal{C}$ is congruence–modular and semi–degenerate, then $\{\alpha/\theta\mid\alpha\in\mathcal{B}(\mathrm{Con}(A))\cap[\theta)\}\subseteq\{(\alpha\vee\theta)/\theta\mid\alpha\in\mathcal{B}(\mathrm{Con}(A))\}\subseteq\mathcal{B}(\mathrm{Con}(A/\theta))$.

Proof: The first inclusion in each statement is trivial. The equalities for principal and on compact congruences follow from Lemma 4, (ii). Now assume that $\mathcal{C}$ is congruence–modular and semi–degenerate, and let $\alpha\in\mathcal{B}(\mathrm{Con}(A))$, so that $\alpha\vee\beta=\nabla_A$ and $[\alpha,\beta]_A=\Delta_A$ for some $\beta\in\mathrm{Con}(A)$, hence, by Lemma 4, (i), and Remark 1, $f(\alpha\vee\mathrm{Ker}(f))\vee f(\beta\vee\mathrm{Ker}(f))=f(\alpha\vee\mathrm{Ker}(f)\vee\beta\vee\mathrm{Ker}(f))=f(\nabla_A)=\nabla_B$ and $[f(\alpha\vee\mathrm{Ker}(f)),f(\beta\vee\mathrm{Ker}(f)]_B=f([\alpha,\beta]_A\vee\mathrm{Ker}(f))=f(\Delta_A)=\Delta_B$, therefore $f(\alpha\vee\mathrm{Ker}(f))\in\mathcal{B}(\mathrm{Con}(B))$. □

Proposition 16 *(i) Assume that f is surjective. Then:*

if $\nabla_A \in \mathrm{PCon}(A)$*, then* $\nabla_B \in \mathrm{PCon}(B)$*, and,*
if $\nabla_A \in \mathcal{K}(A)$*, then* $\nabla_B \in \mathcal{K}(B)$*.*

(ii) $\nabla_A \in \mathrm{PCon}(A)$ *iff* $\nabla_{A/\theta} \in \mathrm{PCon}(A/\theta)$ *for all* $\theta \in \mathrm{Con}(A)$*.*
$\nabla_A \in \mathcal{K}(A)$ *iff* $\nabla_{A/\theta} \in \mathcal{K}(A/\theta)$ *for all* $\theta \in \mathrm{Con}(A)$*.*

Proof: (i) By Lemma 19.
(ii) By (i) for the direct implications, and the fact that A/Δ_A is isomorphic to A, for the converse implications. □

Lemma 20 *If* $\mathcal{C}$ *is congruence–modular, then, for all* $n \in \mathbb{N}^*$ *and any* $\alpha, \beta \in \mathrm{Con}(A)$*:*

(i) if f *is surjective, then*
$[f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]^n_B = f([\alpha, \beta]^n_A \vee \mathrm{Ker}(f))$*;*

(ii) for any $\theta \in \mathrm{Con}(A)$*,* $[(\alpha \vee \theta)/\theta, (\beta \vee \theta)/\theta]^n_{A/\theta} = ([\alpha, \beta]^n_A \vee \theta)/\theta$*;*

(iii) for any $\theta \in \mathrm{Con}(A)$ *and any* $X, Y \in \mathcal{P}(A^2)$*,*
$[Cg_{A/\theta}(X/\theta), Cg_{A/\theta}(Y/\theta)]^n_{A/\theta} = ([Cg_A(X), Cg_A(Y)]^n_A \vee \theta)/\theta$*.*

Proof: (i) We proceed by induction on n. For $n = 1$, this holds by Remark 1. Now take an $n \in \mathbb{N}^*$ such that $[f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]^n_B = f([\alpha, \beta]^n_A \vee \mathrm{Ker}(f))$. Then, by the induction hypothesis and Remark 1, $[f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]^{n+1}_B = [[f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]^n_B, [f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]^n_B]_B = [f([\alpha, \beta]^n_A \vee \mathrm{Ker}(f)), f([\alpha, \beta]^n_A \vee \mathrm{Ker}(f))]_B = f([[\alpha, \beta]^n_A, [\alpha, \beta]^n_A]_A \vee \mathrm{Ker}(f)) = f([\alpha, \beta]^{n+1}_A \vee \mathrm{Ker}(f))$.
(ii) Take $f = p_\theta$ in (i).
(iii) Take $\alpha = Cg_A(X)$ and $\beta = Cg_A(Y)$ in (ii) and apply Lemma 4, (iii). □

We call A a *semiprime algebra* iff $\rho_A(\Delta_A) = \Delta_A$. So A is semiprime iff $\Delta_A \in \mathrm{RCon}(A)$.

Remark 11 *By Lemma 8, if the commutator of* A *equals the intersection, then* A *is semiprime, hence, if* $\mathcal{C}$ *is congruence–distributive, then every member of* $\mathcal{C}$ *is semiprime.*

Proposition 17 $A/\rho_A(\Delta_A)$ *is semiprime.*

Proof: By Lemma 6, $\rho_{A/\rho_A(\Delta_A)}(\Delta_{A/\rho_A(\Delta_A)}) = \rho_A(\rho_A(\Delta_A))/\rho_A(\Delta_A) = \rho_A(\Delta_A)/\rho_A(\Delta_A) = \Delta_{A/\rho_A(\Delta_A)}$. □

Lemma 21 *If A is semiprime, then, for all $\alpha, \beta \in \mathrm{Con}(A)$: $\lambda_A(\alpha) = \mathbf{0}$ iff $\alpha = \Delta_A$, and $[\alpha, \beta]_A = \Delta_A$ iff $\alpha \cap \beta = \Delta_A$.*

Proof: Let $\alpha, \beta \in \mathrm{Con}(A)$. Since $\lambda_A(\Delta_A) = \mathbf{0}$ and $[\alpha, \beta]_A \subseteq \alpha \cap \beta$, the converse implications always hold. Now assume that A is semiprime. If $\lambda_A(\alpha) = \mathbf{0} = \lambda_A(\Delta_A)$, then $\alpha \subseteq \rho_A(\alpha) = \rho_A(\Delta_A) = \Delta_A$, thus $\alpha = \Delta_A$. If $[\alpha, \beta]_A = \Delta_A$, then $\lambda_A(\alpha \cap \beta) = \lambda_A([\alpha, \beta]_A) = \lambda_A(\Delta_A) = \mathbf{0}$, hence $\alpha \cap \beta = \Delta_A$ by the above. □

Lemma 22 *For any $\sigma, \theta \in \mathrm{Con}(A)$: $\theta^\perp = \bigvee\{\alpha \in \mathrm{PCon}(A) \mid [\alpha, \theta]_A = \Delta_A\} = \bigvee\{\alpha \in \mathcal{K}(A) \mid [\alpha, \theta]_A = \Delta_A\} = \bigvee\{\alpha \in \mathrm{Con}(A) \mid [\alpha, \theta]_A = \Delta_A\} = \max\{\alpha \in \mathrm{Con}(A) \mid [\alpha, \theta]_A = \Delta_A\}$, thus: $\sigma \subseteq \theta^\perp$ iff $[\sigma, \theta]_A = \Delta_A$.*

Proof: Let $M = \{\alpha \in \mathrm{Con}(A) \mid [\alpha, \theta]_A = \Delta_A\}$. For all $\alpha \in M$ and all $(a, b) \in \alpha$, $[Cg_A(a, b), \theta]_A \subseteq [\alpha, \theta]_A = \Delta_A$, thus $Cg_A(a, b) \in M \cap \mathrm{PCon}(A)$. Hence $\theta^\perp = \bigvee\limits_{\alpha \in M} \alpha = \bigvee\limits_{\alpha \in M} \bigvee\limits_{(a,b) \in \alpha} Cg_A(a, b) \subseteq \bigvee\limits_{\gamma \in M \cap \mathrm{PCon}(A)} \gamma \subseteq \bigvee\limits_{\gamma \in M \cap \mathcal{K}(A)} \gamma \subseteq \bigvee\limits_{\alpha \in M} \alpha$, therefore $\theta^\perp = \bigvee\limits_{\alpha \in M} \alpha = \bigvee\limits_{\alpha \in M \cap \mathcal{K}(A)} \alpha = \bigvee\limits_{\alpha \in M \cap \mathrm{PCon}(A)} \alpha$. Note, also, that $[\theta^\perp, \theta]_A = [\bigvee\limits_{\alpha \in M} \alpha, \theta]_A = \bigvee\limits_{\alpha \in M} [\alpha, \theta]_A = \bigvee\limits_{\alpha \in M} \Delta_A = \Delta_A$, hence $\theta^\perp \in M$, thus $\theta^\perp = \max(M)$. If $\sigma \subseteq \theta^\perp$, then $[\sigma, \theta]_A \subseteq [\theta^\perp, \theta]_A = \Delta_A$, thus $[\sigma, \theta]_A = \Delta_A$, and conversely: if $[\sigma, \theta]_A = \Delta_A$, then $\sigma \in M$, thus $\sigma \subseteq \max(M) = \theta^\perp$. □

Remark 12 *By Lemma 21, if A is semiprime, then, for any $\theta \in \mathrm{Con}(A)$, $\{\alpha \in \mathrm{Con}(A) \mid [\alpha, \theta]_A = \Delta_A\} = \{\alpha \in \mathrm{Con}(A) \mid \alpha \cap \theta = \Delta_A\}$ is the annihilator of θ in the lattice $\mathrm{Con}(A)$ and it coincides to the principal ideal $(\theta^\perp]$ of $\mathrm{Con}(A)$ by Lemma 22.*

Lemma 23 *For any $\theta \in \mathrm{Con}(A)$, the following hold:*

(i) $\rho_A(\theta) = \bigvee\{\alpha \in \mathrm{Con}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha, \alpha]_A^k \subseteq \theta)\}$
$= \bigvee\{\alpha \in \mathcal{K}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha, \alpha]_A^k \subseteq \theta)\}$
$= \bigvee\{\alpha \in \mathrm{PCon}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha, \alpha]_A^k \subseteq \theta)\}$;

(ii) for any $\alpha \in \mathcal{K}(A)$, $\alpha \subseteq \rho_A(\theta)$ iff there exists a $k \in \mathbb{N}^*$ such that $[\alpha, \alpha]_A^k \subseteq \theta$.

Proof: (i) By Proposition 13 and the fact that $\mathrm{PCon}(A) \subseteq \mathcal{K}(A) \subseteq \mathrm{Con}(A)$, $\rho_A(\theta) = \bigvee\{Cg_A(a,b) \mid (a,b) \in A^2, (\exists\, k \in \mathbb{N}^*)\,([Cg_A(a,b), Cg_A(a,b)]^k_A \subseteq \theta)\} = \bigvee\{\alpha \in \mathrm{PCon}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A \subseteq \theta)\} \subseteq \bigvee\{\alpha \in \mathcal{K}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A \subseteq \theta)\} \subseteq \bigvee\{\alpha \in \mathrm{Con}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A \subseteq \theta)\} \subseteq \bigvee\{\alpha \in \mathrm{PCon}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A \subseteq \theta)\}$, where the last inclusion holds because, for any $\alpha \in \mathrm{Con}(A)$, if $k \in \mathbb{N}^*$ is such that $[\alpha,\alpha]^k_A \subseteq \theta$, then, for any $(a,b) \in \alpha$, $[Cg_A(a,b), Cg_A(a,b)]^k_A \subseteq [\alpha,\alpha]^k_A \subseteq \theta$, thus $\alpha = \bigvee_{(a,b)\in\alpha} Cg_A(a,b) \subseteq \bigvee\{Cg_A(a,b) \mid (a,b) \in A^2, (\exists\, k \in \mathbb{N}^*)\,([Cg_A(a,b), Cg_A(a,b)]^k_A \subseteq \theta)\}$. Hence the equalities in the enunciation.
(ii) The converse implication follows directly from (i).

For the direct implication, from (i) it follows that, for any $\alpha \in \mathcal{K}(A)$ such that $\alpha \subseteq \rho_A(\theta)$, there exist non–empty families $(\beta_j)_{j\in J} \subseteq \mathcal{K}(A)$ and $(k_j)_{j\in J} \subseteq \mathbb{N}^*$ such that $\alpha \subseteq \bigvee_{j\in J} \beta_j$ and $[\beta_j, \beta_j]^{k_j}_A \subseteq \theta$ for all $j \in J$. Since $\alpha \in \mathcal{K}(A)$, it follows that there exist an $n \in \mathbb{N}^*$ and $j_1, \ldots, j_n \in J$ such that $\alpha \subseteq \bigvee_{i=1}^{n} \beta_{j_i}$. Let $j = \max\{j_1, \ldots, j_n\} \in \mathbb{N}^*$. Then $[\beta_{j_i}, \beta_{j_i}]^k_A \subseteq [\beta_{j_i}, \beta_{j_i}]^{k_i}_A \subseteq \theta$ for each $i \in \overline{1,n}$, thus, by Lemma 16, (vi), $[\alpha,\alpha]^{k^n}_A \subseteq [\bigvee_{i=1}^{n} \beta_{j_i}, \bigvee_{i=1}^{n} \beta_{j_i}]^{k^n}_A \subseteq \bigvee_{i=1}^{n} [\beta_{j_i}, \beta_{j_i}]^k_A \subseteq \theta$. □

Proposition 18 *(i)* $\rho_A(\Delta_A) = \bigvee\{\alpha \in \mathrm{Con}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A = \Delta_A)\}$
$= \bigvee\{\alpha \in \mathcal{K}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A = \Delta_A)\}$
$= \bigvee\{\alpha \in \mathrm{PCon}(A) \mid (\exists\, k \in \mathbb{N}^*)\,([\alpha,\alpha]^k_A = \Delta_A)\}$*;*

(ii) for any $\alpha \in \mathcal{K}(A)$, $\alpha \subseteq \rho_A(\Delta_A)$ *iff there exists a* $k \in \mathbb{N}^*$ *such that* $[\alpha,\alpha]^k_A = \Delta_A$.

Proof: By Lemma 23. □

Corollary 2 *A is semiprime iff, for any* $\alpha \in \mathcal{K}(A)$ *and any* $k \in \mathbb{N}^*$*, if* $[\alpha,\alpha]^k_A = \Delta_A$*, then* $\alpha = \Delta_A$.

Throughout the rest of this section, we shall assume that $[\theta, \nabla_A]_A = \theta$ for all $\theta \in \mathrm{Con}(A)$.

Lemma 24 *$(\mathcal{B}(\mathrm{Con}(A)), \vee, [\cdot,\cdot]_A = \cap, \perp, \Delta_A, \nabla_A)$ is a Boolean algebra.*

Proof: We follow, in part, the argument from [27, Lemma 4]. Let $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$, so that there exist $\overline{\alpha}, \overline{\beta} \in \mathcal{B}(\mathrm{Con}(A))$ such that $\alpha \vee \overline{\alpha} = \beta \vee \overline{\beta} = \nabla_A$ and $\alpha \cap \overline{\alpha} = \beta \cap \overline{\beta} = \Delta_A$. Then, by Remark 10, the following hold: $(\alpha\vee\beta)\cap\overline{\alpha}\cap\overline{\beta} = (\alpha\cap\overline{\alpha}\cap\overline{\beta})\vee(\beta\cap\overline{\alpha}\cap\overline{\beta}) = \Delta_A\vee\Delta_A = \Delta_A$ and, since $\overline{\alpha}\cap\beta \subseteq \beta$, it follows that $\alpha\vee\beta\vee(\overline{\alpha}\cap\overline{\beta}) = \alpha\vee\beta\vee(\overline{\alpha}\cap\beta)\vee(\overline{\alpha}\cap\overline{\beta}) = \alpha\vee\beta\vee(\overline{\alpha}\cap(\beta\vee\overline{\beta})) = \alpha\vee\beta\vee(\overline{\alpha}\cap\nabla_A) = \alpha\vee\beta\vee\overline{\alpha} = \nabla_A$. Analogously, $(\overline{\alpha}\vee\overline{\beta})\cap\alpha\cap\beta = \Delta_A$ and $\overline{\alpha}\vee\overline{\beta}\vee(\alpha\cap\beta) = \nabla_A$. Hence $\alpha\vee\beta, \alpha\cap\beta \in \mathcal{B}(\mathrm{Con}(A))$. Clearly, $\Delta_A, \nabla_A \in \mathcal{B}(\mathrm{Con}(A))$. Therefore $\mathcal{B}(\mathrm{Con}(A))$ is a bounded sublattice of $\mathrm{Con}(A)$. By Remark 10, it follows that $(\mathcal{B}(\mathrm{Con}(A)), \vee, [\cdot,\cdot]_A = \cap, \Delta_A, \nabla_A)$ is a bounded distributive lattice, and, by its definition, it is also complemented, thus it is a Boolean lattice. By a well–known characterization of the complement in a Boolean lattice, for any $\theta \in \mathcal{B}(\mathrm{Con}(A))$, the complement of θ in $\mathcal{B}(\mathrm{Con}(A))$ is $\overline{\theta} = \max\{\alpha\in\mathcal{B}(\mathrm{Con}(A)) \mid \alpha\cap\theta=\Delta_A\} = \max\{\alpha\in\mathcal{B}(\mathrm{Con}(A)) \mid [\alpha,\theta]_A=\Delta_A\} \subseteq \max\{\alpha \in \mathrm{Con}(A) \mid [\alpha,\theta]_A = \Delta_A\} = \theta^\perp$ according to Lemma 22, thus $\nabla_A = \theta \vee \overline{\theta} \subseteq \theta \vee \theta^\perp$, so $\theta \vee \theta^\perp = \nabla_A$. Again by Lemma 22, $\Delta_A = [\theta, \theta^\perp]_A = \theta \cap \theta^\perp$. Therefore $\theta^\perp \in \mathcal{B}(\mathrm{Con}(A))$ and $\theta^\perp$ is the complement of θ in $\mathcal{B}(\mathrm{Con}(A))$. □

Proposition 19
- *$\mathcal{B}(\mathrm{Con}(A)) \subseteq \mathcal{K}(A)$.*
- *$\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/\equiv_A \subseteq \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{B}(\mathrm{Con}(A)/\equiv_A)$ and $\lambda_A\mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ is a Boolean morphism.*
- *If $\mathcal{K}(A) = \mathcal{B}(\mathrm{Con}(A))$, then $\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$.*

Proof: Let $\alpha \in \mathcal{B}(\mathrm{Con}(A))$, so that $\alpha \vee \beta = \nabla_A$ and $\alpha \cap \beta = \Delta_A$ for some $\beta \in \mathrm{Con}(A)$.

Now let $\emptyset \neq (\alpha_i)_{i\in I} \subseteq \mathrm{Con}(A)$ such that $\alpha \subseteq \bigvee_{i\in I} \alpha_i$, so that $\beta \vee \bigvee_{i\in I} \alpha_i = \nabla_A \in \mathcal{K}(A)$, thus $\nabla_A = \beta \vee \bigvee_{j=1}^{n} \alpha_{i_j}$ for some $n \in \mathbb{N}^*$ and some $i_1, \ldots, i_n \subseteq I$, hence, by Lemma 18, (i), $\alpha = [\alpha, \nabla_A]_A = [\alpha, \beta \vee \bigvee_{j=1}^{n} \alpha_{i_j}]_A = [\alpha, \beta]_A \vee [\alpha, \bigvee_{j=1}^{n} \alpha_{i_j}]_A = \Delta_A \vee [\alpha, \bigvee_{j=1}^{n} \alpha_{i_j}]_A = [\alpha, \bigvee_{j=1}^{n} \alpha_{i_j}]_A \subseteq \bigvee_{j=1}^{n} \alpha_{i_j}$, hence $\alpha \in \mathcal{K}(A)$.

$\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/\equiv_A$. Since $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A)) \subseteq \mathcal{K}(A)$, we have $\lambda_A(\alpha) \in \mathcal{L}(A)$. $\mathbf{1} = \lambda_A(\nabla_A) = \lambda_A(\alpha \vee \beta) = \lambda_A(\alpha) \vee \lambda_A(\beta)$ and

$\mathbf{0} = \lambda_A(\Delta_A) = \lambda_A(\alpha \cap \beta) = \lambda_A(\alpha) \wedge \lambda_A(\beta)$, hence $\lambda_A(\alpha) \in \mathcal{B}(\mathcal{L}(A))$. Therefore $\lambda_A(\mathcal{B}(\mathrm{Con}(A))) \subseteq \mathcal{B}(\mathcal{L}(A))$. Since $\mathcal{L}(A)$ is a bounded sublattice of the bounded distributive lattice $\mathrm{Con}(A)/\equiv_A$, it follows that $\mathcal{B}(\mathcal{L}(A))$ is a Boolean subalgebra of $\mathcal{B}(\mathrm{Con}(A)/\equiv_A)$. Hence $\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/\equiv_A \subseteq \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{B}(\mathrm{Con}(A)/\equiv_A)$. $\lambda_A : \mathrm{Con}(A) \rightarrow \mathrm{Con}(A)/\equiv_A$ is a (surjective) bounded lattice morphism. Hence $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \rightarrow \mathcal{B}(\mathcal{L}(A))$ is well defined and it is a bounded lattice morphism, thus it is a Boolean morphism.

If $\mathcal{K}(A) = \mathcal{B}(\mathrm{Con}(A))$, then $\mathcal{L}(A) = \lambda_A(\mathcal{K}(A)) = \lambda_A(\mathcal{B}(\mathrm{Con}(A))) \subseteq \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ by the above, thus $\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$. □

Throughout the rest of this section, $\mathcal{C}$ shall be congruence–modular and semi–degenerate.

Theorem 5 *(i) The Boolean morphism* $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \rightarrow \mathcal{B}(\mathcal{L}(A))$ *is injective.*

(ii) If the commutator of A is associative, then $\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathcal{L}(A)) = \mathcal{B}(\mathrm{Con}(A))/\equiv_A \subseteq \mathcal{B}(\mathrm{Con}(A)/\equiv_A)$ *and* $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \rightarrow \mathcal{B}(\mathcal{L}(A))$ *is a Boolean isomorphism.*

(iii) If A is semiprime, then $\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathcal{L}(A)) = \mathcal{B}(\mathrm{Con}(A))/\equiv_A = \mathcal{B}(\mathrm{Con}(A)/\equiv_A)$ *and* $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \rightarrow \mathcal{B}(\mathcal{L}(A))$ *is a Boolean isomorphism.*

Proof: (i) By Proposition 19, $\lambda_A \mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \rightarrow \mathcal{B}(\mathcal{L}(A))$ is a Boolean morphism. $\lambda_A(\alpha) = \mathbf{1}$ iff $\alpha = \nabla_A$, hence this Boolean morphism is injective.

(ii) Assume that the commutator of A is associative, and let $x \in \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{L}(A) = \lambda_A(\mathcal{K}(A))$, so that $x \vee y = \mathbf{1}$ and $x \wedge y = \mathbf{0}$ for some $y \in \mathcal{B}(\mathcal{L}(A))$ and there exist $\alpha, \beta \in \mathcal{K}(A)$ such that $x = \lambda_A(\alpha)$ and $y = \lambda_A(\beta)$. Then $\lambda_A(\alpha \vee \beta) = \lambda_A(\alpha) \vee \lambda_A(\beta) = x \vee y = \mathbf{1} = \lambda_A(\nabla_A)$, hence $\alpha \vee \beta = \nabla_A$. We also have $\lambda_A([\alpha, \beta]_A) = \lambda_A(\alpha) \wedge \lambda_A(\beta) = x \wedge y = \mathbf{0} = \lambda_A(\Delta_A)$, thus $[\alpha, \beta]_A \subseteq \rho_A(\alpha \cap \beta) = \rho_A(\Delta_A)$, and, since $\mathcal{K}(A)$ is closed with respect to the commutator, we have $[\alpha, \beta]_A \in \mathcal{K}(A)$, thus, according to Proposition 18, (ii), $[[\alpha, \alpha]_A^k, [\beta, \beta]_A^k]_A = [\alpha, \beta]_A^{k+1} = [[\alpha, \beta]_A, [\alpha, \beta]_A]_A^k = \Delta_A$ for some $k \in \mathbb{N}^*$; we have applied Lemma 15. But $\alpha \vee \beta = \nabla_A$, thus $[\alpha, \alpha]_A^k \vee [\beta, \beta]_A^k = \nabla_A$, hence $[\alpha, \alpha]_A^k \cap [\beta, \beta]_A^k = [[\alpha, \alpha]_A^k, [\beta, \beta]_A^k]_A = \Delta_A$ by Lemma 18, (iii) and (i). Therefore $[\alpha, \alpha]_A^k \in \mathcal{B}(\mathrm{Con}(A))$, thus $x = \lambda_A(\alpha) = \lambda_A([\alpha, \alpha]_A^k) \in \lambda_A(\mathcal{B}(\mathrm{Con}(A)))$, hence $\mathcal{B}(\mathcal{L}(A)) \subseteq \lambda_A(\mathcal{B}(\mathrm{Con}(A)))$, thus $\mathcal{B}(\mathcal{L}(A)) \subseteq \lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/\equiv_A \subseteq \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{B}(\mathrm{Con}(A)/\equiv_A)$ by Proposition 19, therefore

$\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/_{\equiv_A} = \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{B}(\mathrm{Con}(A)/_{\equiv_A})$. Therefore $\lambda_A\mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ is surjective, so, by (i), it is a Boolean isomorphism.

(iii) Assume that A is semiprime, and let $x \in \mathcal{B}(\mathrm{Con}(A)/_{\equiv_A})$, so that $x \vee y = \mathbf{1}$ and $x \wedge y = \mathbf{0}$ for some $y \in \mathcal{B}(\mathrm{Con}(A)/_{\equiv_A})$. Hence there exist $\alpha, \beta \in \mathrm{Con}(A)$ such that $x = \lambda_A(\alpha)$ and $y = \lambda_A(\beta)$, thus $\mathbf{1} = x \vee y = \lambda_A(\alpha) \vee \lambda_A(\beta) = \lambda_A(\alpha \vee \beta)$ and $\mathbf{0} = x \wedge y = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A(\alpha \cap \beta)$, therefore $\alpha \vee \beta = \nabla_A$ and $\alpha \cap \beta = \Delta_A$ by Lemma 21. Hence $\alpha \in \mathcal{B}(\mathrm{Con}(A))$, thus $x = \lambda_A(\alpha) \in \lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/_{\equiv_A}$, therefore, by Proposition 19, $\mathcal{B}(\mathrm{Con}(A)/_{\equiv_A}) \subseteq \lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/_{\equiv_A} \subseteq \mathcal{B}(\mathcal{L}(A)) \subseteq \mathcal{B}(\mathrm{Con}(A)/_{\equiv_A})$, hence $\lambda_A(\mathcal{B}(\mathrm{Con}(A))) = \mathcal{B}(\mathrm{Con}(A))/_{\equiv_A} = \mathcal{B}(\mathcal{L}(A)) = \mathcal{B}(\mathrm{Con}(A)/_{\equiv_A})$. Therefore $\lambda_A\mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ is surjective, so, by (i), it is a Boolean isomorphism. □

Lemma 25 *If A is semiprime and $\alpha \in \mathrm{Con}(A)$, then: $\alpha \in \mathcal{B}(\mathrm{Con}(A))$ iff $\lambda_A(\alpha) \in \mathcal{B}(\mathcal{L}(A))$.*

Proof: We apply Proposition 19, which, first of all, gives us the direct implication. For the converse, assume that $\lambda_A(\alpha) \in \mathcal{B}(\mathcal{L}(A)) = \mathcal{B}(\mathrm{Con}(A)/_{\equiv_A})$, so that there exists a $\beta \in \mathrm{Con}(A)$ with $\lambda_A(\alpha \vee \beta) = \lambda_A(\alpha) \vee \lambda_A(\beta) = \mathbf{1} = \lambda_A(\nabla_A)$ and $\lambda_A(\alpha \cap \beta) = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \mathbf{0}$, thus $\alpha \vee \beta = \nabla_A$ and $\alpha \cap \beta = \Delta_A$ by Lemma 21. Therefore $\alpha \in \mathcal{B}(\mathrm{Con}(A))$. □

For any $\Omega \subseteq \mathrm{Con}(A)$, let us consider the property:

(A, Ω) for all $\alpha, \beta \in \Omega$ and all $n \in \mathbb{N}^*$, there exists a $k \in \mathbb{N}^*$ such that $[[\alpha, \alpha]^k_A, [\beta, \beta]^k_A]_A \subseteq [\alpha, \beta]^n_A$

Remark 13 *By Lemma 15, if the commutator of A is associative, then $(A, \mathrm{Con}(A))$ holds.*

Notice, from the proof of statement (iii) from Theorem 5, that this statement, and thus the fact that $\lambda_A\mid_{\mathcal{B}(\mathrm{Con}(A))}: \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A))$ is a Boolean isomorphism, also hold if property $(A, \mathcal{K}(A))$ is fulfilled, instead of the associativity of the commutator of A.

Open problem 1 *Under the current context, determine whether $(A, \mathcal{K}(A))$ always holds; if it doesn't, then determine whether $(A, \mathcal{K}(A))$ is equivalent to the associativity of the commutator of A.*

Throughout the rest of this paper, all annihilators shall be considerred in the bounded distributive lattice $\mathcal{L}(A)$, so they shall be ideals of the lattice $\mathcal{L}(A)$. Recall that $\mathcal{L}(A) = \lambda_A(\mathcal{K}(A))$.

Lemma 26 *For any* $\alpha \in \mathcal{K}(A)$*:* $\mathrm{Ann}(\alpha^*) = \{\lambda_A(\beta) \mid \beta \in \mathcal{K}(A), \lambda_A([\alpha,\beta]_A) = \mathbf{0}\}$*, and, if* A *is semiprime, then* $\mathrm{Ann}(\alpha^*) = \{\lambda_A(\beta) \mid \beta \in \mathcal{K}(A), [\alpha,\beta]_A = \Delta_A\}$*.*

Proof: By Lemma 10, $\mathrm{Ann}(\alpha^*) = \mathrm{Ann}((\lambda_A(\alpha)]) = \{\lambda_A(\beta) \mid \beta \in \mathcal{K}(A), (\forall\, x \in (\lambda_A(\alpha)])\, (x \wedge \lambda_A(\beta) = \mathbf{0})\} = \{\lambda_A(\beta) \mid \beta \in \mathcal{K}(A), \lambda_A(\alpha) \wedge \lambda_A(\beta) = \mathbf{0}\} = \{\lambda_A(\beta) \mid \beta \in \mathcal{K}(A), \lambda_A([\alpha,\beta]_A) = \mathbf{0}\}$. By Lemma 21, if A is semiprime, then, for any $\beta \in \mathcal{K}(A)$, $\lambda_A([\alpha,\beta]_A) = \mathbf{0}$ iff $[\alpha,\beta]_A = \Delta_A$, hence the second equality in the enunciation. $\square$

Lemma 27 *For any* $\alpha \in \mathrm{Con}(A)$ *and any* $I \in \mathrm{Id}(\mathcal{L}(A))$*, if* $\mathrm{Ann}(\alpha^*) \subseteq I$*, then* $\alpha^\perp \subseteq I_*$*. If* A *is semiprime and* $\alpha \in \mathcal{K}(A)$*, then the converse implication holds, as well.*

Proof: For the direct implication, assume that $\mathrm{Ann}(\alpha^*) \subseteq I$ and let $\beta \in \mathcal{K}(A)$ such that $[\alpha,\beta]_A = \Delta_A$, hence $\lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A([\alpha,\beta]_A) = \lambda_A(\Delta_A) = \mathbf{0}$. Now let $x \in \alpha^*$, so that $x = \lambda_A(\gamma)$ for some $\gamma \in \mathcal{K}(A)$ with $\gamma \subseteq \alpha$. Then $x = \lambda_A(\gamma) \leq \lambda_A(\alpha)$, hence $x \wedge \lambda_A(\beta) = \lambda_A(\gamma) \wedge \lambda_A(\beta) \leq \lambda_A(\alpha) \wedge \lambda_A(\beta) = \mathbf{0}$, so $x \wedge \lambda_A(\beta) = \mathbf{0}$, thus $\lambda_A(\beta) \in \mathrm{Ann}(\alpha^*) \subseteq I$, therefore $\beta \subseteq I_*$ by Lemma 11, (ii). According to Lemma 22, $\alpha^\perp = \bigvee\{\beta \in \mathcal{K}(A) \mid [\alpha,\beta]_A = \Delta_A\} \subseteq I_*$.

For the converse implication, assume that A is semiprime, $\alpha \in \mathcal{K}(A)$ and $\alpha^\perp \subseteq I_*$, and let $x \in \mathrm{Ann}(\alpha^*)$, which means that $x = \lambda_A(\beta)$ for some $\beta \in \mathcal{K}(A)$ with $[\alpha,\beta]_A = \Delta_A$, according to Lemma 26. Hence, by Lemma 22 and Lemma 11, (ii), $\beta \subseteq \alpha^\perp \subseteq I_*$, thus $x = \lambda_A(\beta) \in I$, therefore $\mathrm{Ann}(\alpha^\perp) \subseteq I$. $\square$

Theorem 6 *(i) For any* $\theta \in \mathrm{Con}(A)$*:* $(\theta^\perp)^* \subseteq \mathrm{Ann}(\theta^*)$*, and, if* A *is semiprime, then* $(\theta^\perp)^* = \mathrm{Ann}(\theta^*)$*.*

(ii) For any $I \in \mathrm{Id}(\mathcal{L}(A))$*:* $(I_*)^\perp \subseteq \mathrm{Ann}(I)_*$*, and, if* A *is semiprime, then* $(I_*)^\perp = \mathrm{Ann}(I)_*$*.*

Proof: (i) $(\theta^\perp)^* = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), \alpha \subseteq \theta^\perp\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), [\alpha,\theta]_A = \Delta_A\}$, by Lemma 22. $\mathrm{Ann}(\theta^*) = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), (\forall\, x \in \theta^*)\, (\lambda_A(\alpha) \wedge x = \lambda_A(\Delta_A))\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), (\forall\, \beta \in \mathcal{K}(A))\, (\beta \subseteq \theta \Rightarrow \lambda_A([\alpha,\beta]_A) = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A(\Delta_A))\} = \{\lambda_A(\alpha) \mid \alpha \in \mathcal{K}(A), (\forall\, \beta \in \mathcal{K}(A))\, (\beta \subseteq \theta \Rightarrow \rho_A([\alpha,\beta]_A) = \rho_A(\Delta_A))\}$.

Let $\alpha \in \mathcal{K}(A)$ such that $\lambda_A(\alpha) \in (\theta^\perp)^*$, which means that $[\alpha,\theta]_A = \Delta_A$. Then, for any $\beta \in \mathcal{K}(A)$ fulfilling $\beta \subseteq \theta$, we have $[\alpha,\beta]_A \subseteq [\alpha,\theta]_A = \Delta_A$, so $[\alpha,\beta]_A=\Delta_A$, thus $\rho_A([\alpha,\beta]_A)=\rho_A(\Delta_A))$, hence $\lambda_A(\alpha)\in\mathrm{Ann}(\theta^*)$. Therefore $(\theta^{\perp})^*\subseteq\mathrm{Ann}(\theta^*)$.

Now assume that A is semiprime and $\alpha\in\mathcal{K}(A)$ such that $\lambda_A(\alpha)\in\mathrm{Ann}(\theta^*)$, which means that, for all $\beta\in\mathcal{K}(A)$ such that $\beta\subseteq\theta$, $[\alpha,\beta]_A\subseteq\rho_A([\alpha,\beta]_A)=\rho_A(\Delta_A)=\Delta_A$, so $[\alpha,\beta]_A=\Delta_A$. $\theta=\bigvee\limits_{(a,b)\in\theta}Cg_A(a,b)\subseteq\bigvee\{\beta\in\mathcal{K}(A)\mid\beta\subseteq\theta\}\subseteq\theta$, thus $\theta=\bigvee\{\beta\in\mathcal{K}(A)\mid\beta\subseteq\theta\}$, so $[\alpha,\theta]_A=[\alpha,\bigvee\{\beta\in\mathcal{K}(A)\mid\beta\subseteq\theta\}]_A=\bigvee\{[\alpha,\beta]_A\mid\beta\in\mathcal{K}(A),\beta\subseteq\theta\}=\bigvee\{\Delta_A\mid\beta\in\mathcal{K}(A),\beta\subseteq\theta\}=\bigvee\{\Delta_A\}=\Delta_A$, therefore $\lambda_A(\alpha)\in(\theta^{\perp})^*$, hence $\mathrm{Ann}(\theta^*)\subseteq(\theta^{\perp})^*$, thus $\mathrm{Ann}(\theta^*)=(\theta^{\perp})^*$.

(ii) $(I_*)^{\perp}=\bigvee\{\alpha\in\mathcal{K}(A)\mid[\alpha,I_*]_A=\Delta_A\}=\bigvee\{\alpha\in\mathcal{K}(A)\mid[\alpha,\bigvee\{\beta\in\mathcal{K}(A)\mid\lambda_A(\beta)\in I\}]_A=\Delta_A\}=\bigvee\{\alpha\in\mathcal{K}(A)\mid\bigvee\{[\alpha,\beta]_A\in\mathcal{K}(A)\mid\beta\in\mathcal{K}(A),\lambda_A(\beta)\in I\}=\Delta_A\}=\bigvee\{\alpha\in\mathcal{K}(A)\mid(\forall\,\beta\in\mathcal{K}(A))\,(\lambda_A(\beta)\in I\Rightarrow[\alpha,\beta]_A=\Delta_A)\}$. $(\mathrm{Ann}(I))_*=\bigvee\{\alpha\in\mathcal{K}(A)\mid\lambda_A(\alpha)\in\mathrm{Ann}(I)\}=\bigvee\{\alpha\in\mathcal{K}(A)\mid(\forall\,\beta\in\mathcal{K}(A))\,(\lambda_A(\beta)\in I\Rightarrow\lambda_A(\alpha)\wedge\lambda_A(\beta)=\mathbf{0})\}=\bigvee\{\alpha\in\mathcal{K}(A)\mid(\forall\,\beta\in\mathcal{K}(A))\,(\lambda_A(\beta)\in I\Rightarrow\lambda_A([\alpha,\beta]_A)=\lambda_A(\Delta_A))\}$.

For all $\alpha,\beta\in\mathrm{Con}(A)$, if $[\alpha,\beta]_A=\Delta_A$, then $\lambda_A([\alpha,\beta]_A)=\lambda_A(\Delta_A)$, hence $(I_*)^{\perp}\subseteq\mathrm{Ann}(I)_*$.

By Lemma 21, if A is semiprime, then, for every $\alpha,\beta\in\mathrm{Con}(A)$, $\lambda_A([\alpha,\beta]_A)=\lambda_A(\Delta_A)=\mathbf{0}$ iff $[\alpha,\beta]_A=\Delta_A$, hence $(I_*)^{\perp}=\mathrm{Ann}(I)_*$. □

We call A a *hyperarchimedean algebra* iff, for all $\alpha\in\mathrm{PCon}(A)$, there exists an $n\in\mathbb{N}^*$ such that $[\alpha,\alpha]_A^n\in\mathcal{B}(\mathrm{Con}(A))$.

Remark 14 *If $\alpha\in\mathrm{Con}(A)$ and $n\in\mathbb{N}^*$ are such that $[\alpha,\alpha]_A^n\in\mathcal{B}(\mathrm{Con}(A))$, then, by Remark 10, $[\alpha,\alpha]_A^{n+1}=[[\alpha,\alpha]_A^n,[\alpha,\alpha]_A^n]_A=[\alpha,\alpha]_A^n\cap[\alpha,\alpha]_A^n=[\alpha,\alpha]_A^n$, thus $[\alpha,\alpha]_A^k=[\alpha,\alpha]_A^n$ for all $k\in\mathbb{N}$ such that $k\geq n$.*

Remark 15 *If $[\alpha,\alpha]_A\in\mathcal{B}(\mathrm{Con}(A))$ for all $\alpha\in\mathrm{PCon}(A)$, then A is hyperarchimedean. Thus, if $\mathrm{PCon}(A)\subseteq\mathcal{B}(\mathrm{Con}(A))$ and A has principal commutators, then A is hyperarchimedean. If the commutator of A equals the intersection, for instance if $\mathcal{C}$ is congruence–distributive, then: A is hyperarchimedean iff $\mathrm{PCon}(A)\subseteq\mathcal{B}(\mathrm{Con}(A))$. By Lemma 24 and Proposition 19, the following equivalences hold: $\mathrm{PCon}(A)\subseteq\mathcal{B}(\mathrm{Con}(A))$ iff $\mathcal{K}(A)\subseteq\mathcal{B}(\mathrm{Con}(A))$ iff $\mathcal{K}(A)=\mathcal{B}(\mathrm{Con}(A))$.*

Remark 16 *By Lemma 24, the lattice $\mathrm{Con}(A)$ is Boolean iff $\mathrm{Con}(A)=\mathcal{B}(\mathrm{Con}(A))$, which implies that the commutator of A equals the intersection, according to Remark 10, and thus, since $\mathrm{PCon}(A)\subseteq\mathrm{Con}(A)=$*

$\mathcal{B}(\mathrm{Con}(A))$, A is hyperarchimedean, while Remark 11 ensures us that A is semiprime. From Proposition 19, we obtain the following equivalences: $\mathrm{Con}(A)$ is a Boolean lattice iff $\mathcal{B}(\mathrm{Con}(A)) = \mathrm{Con}(A)$ iff $\mathcal{B}(\mathrm{Con}(A)) = \mathcal{K}(A) = \mathrm{Con}(A)$. Of course, since $\mathcal{L}(A)$ is a bounded distributive lattice, $\mathcal{L}(A)$ is a Boolean algebra iff $\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$.

Proposition 20 *(i) If A is semiprime, then: $\mathcal{K}(A) = \mathcal{B}(\mathrm{Con}(A))$ iff $\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$.*

(ii) If $\mathrm{Con}(A)$ is a Boolean lattice, then A is hyperarchimedean and semiprime and $\mathcal{L}(A)$ is isomorphic to $\mathrm{Con}(A)$, in particular $\mathcal{L}(A)$ is a Boolean lattice, as well.

Proof: (i) The direct implication holds by Proposition 19. For the converse, let $\alpha \in \mathcal{K}(A)$, so that $\lambda_A(\alpha) \in \mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$, thus $\alpha \in \mathcal{B}(\mathrm{Con}(A))$ by Lemma 25. Hence $\mathcal{K}(A) \subseteq \mathcal{B}(\mathrm{Con}(A))$, thus $\mathcal{K}(A) = \mathcal{B}(\mathrm{Con}(A))$.
(ii) By Remark 16, A is hyperarchimedean and semiprime, and $\mathcal{B}(\mathrm{Con}(A)) = \mathcal{K}(A) = \mathrm{Con}(A)$, hence $\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$ by (i), and thus $\lambda_A : \mathrm{Con}(A) = \mathcal{B}(\mathrm{Con}(A)) \to \mathcal{B}(\mathcal{L}(A)) = \mathcal{L}(A)$ is a Boolean isomorphism, according to Theorem 5, (iii). □

Proposition 21 *If A is hyperarchimedean, then A/θ is hyperarchimedean for all $\theta \in \mathrm{Con}(A)$.*

Proof: Let $\theta \in \mathrm{Con}(A)$. For any $a, b \in A$, there exists an $n \in \mathbb{N}^*$ such that $[Cg_A(a,b), Cg_A(a,b)]_A^n \in \mathcal{B}(\mathrm{Con}(A))$. Then, according to Lemma 20, (iii), and Lemma 19, $[Cg_{A/\theta}(a/\theta, b/\theta), Cg_{A/\theta}(a/\theta, b/\theta)]_{A/\theta}^n = ([Cg_A(a,b), Cg_A(a,b)]_A^n \vee \theta)/\theta \in \mathcal{B}(\mathrm{Con}(A/\theta))$, therefore A/θ is hyperarchimedean. □

Now let us give a direct proof for part of Theorem 8 in the next section:

Proposition 22 *If A is hyperarchimedean, then $\mathcal{L}(A)$ is a Boolean lattice.*

Proof: Let $\theta \in \mathcal{K}(A)$, so that $\theta = \alpha_1 \vee \ldots \vee \alpha_n$ for some $n \in \mathbb{N}^*$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{PCon}(A)$. Since A is hyperarchimedean, there exists a $k \in \mathbb{N}^*$ such that, for all $i \in \overline{1,n}$, $[\alpha_i, \alpha_i]_A^k \in \mathcal{B}(\mathrm{Con}(A))$, thus $\lambda_A(\alpha_i) = \lambda_A([\alpha_i, \alpha_i]_A^k) \in \lambda_A(\mathcal{B}(\mathrm{Con}(A))) \subseteq \mathcal{B}(\mathcal{L}(A))$ by Proposition 19, so that $\lambda_A(\theta) = \lambda_A(\alpha_1) \vee \ldots \vee \lambda_A(\alpha_n) \in \mathcal{B}(\mathcal{L}(A))$. Hence $\lambda_A(\mathcal{K}(A)) = \mathcal{L}(A) \subseteq \mathcal{B}(\mathcal{L}(A))$, thus $\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))$, so $\mathcal{L}(A)$ is a Boolean lattice. □

7 A Reticulation Functor

Throughout this section, $\mathcal{C}$ shall be congruence–modular and semi–degenerate and such that, in each of its members, the set of the compact congruences is closed w.r.t. the commutator. Also, the morphism $f : A \to B$ shall be surjective, so that the map $\varphi_f : \mathrm{Con}(A) \to \mathrm{Con}(B)$, $\varphi_f(\alpha) = f(\alpha \vee \mathrm{Ker}(f))$ for all $\alpha \in \mathrm{Con}(A)$, is well defined.

Remark 17 *By Lemma 19, $\varphi_f(\mathcal{K}(A)) = \mathcal{K}(B)$.*

For any algebra M from $\mathcal{C}$ and any $X \subseteq M^2$, let us denote $V_M(X) = V_M(Cg_M(X))$. Then, by the proof of [1, Proposition 2.1] and Lemma 4, (i), for all $\alpha \in \mathrm{Con}(A)$, $\{f(\phi) \mid \phi \in V_A(\alpha)\} = f(V_A(\alpha)) = V_B(f(\alpha)) = V_B(Cg_B(f(\alpha))) = V_B(f(\alpha \vee \mathrm{Ker}(f))) = V_B(\varphi_f(\alpha))$.

$$\begin{array}{ccc}
\mathrm{Con}(A) & \xrightarrow{\varphi_f} & \mathrm{Con}(B) \\
\cup\!| & & \cup\!| \\
\mathcal{K}(A) & \xrightarrow{\varphi_f} & \mathcal{K}(B) \\
\lambda_A \downarrow & & \downarrow \lambda_B \\
\mathcal{L}(A) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(B)
\end{array}$$

Let us define $\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)$, for all $\alpha \in \mathcal{K}(A)$, $\mathcal{L}(f)(\widehat{\alpha}) = \widehat{\varphi_f(\alpha)}$, that is $\mathcal{L}(f)(\lambda_A(\alpha)) = \lambda_B(f(\alpha \vee \mathrm{Ker}(f)))$.

Proposition 23 *$\mathcal{L}(f)$ is well defined and it is a surjective lattice morphism.*

Proof: By Remark 17, the restriction $\varphi_f \mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ is well defined and surjective. Let $\alpha, \beta \in \mathcal{K}(A)$ such that $\lambda_A(\alpha) = \lambda_A(\beta)$, so that $\rho_A(\alpha) = \rho_A(\beta)$, thus $V_A(\alpha) = V_A(\beta)$, hence $V_B(\varphi_f(\alpha)) = f(V_A(\alpha)) = f(V_A(\beta)) = V_B(\varphi_f(\beta))$, thus $\rho_B(\varphi_f(\alpha)) = \rho_B(\varphi_f(\beta))$, so $\lambda_B(\varphi_f(\alpha)) = \lambda_B(\varphi_f(\beta))$, that is $\mathcal{L}(f)(\lambda_A(\alpha)) = \mathcal{L}(f)(\lambda_A(\beta))$; we have used Lemma 6, (i), and Remark 17. Hence $\mathcal{L}(f)$ is well defined. $\lambda_B : \mathcal{K}(B) \to \mathcal{L}(B)$ $\varphi_f \mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$ are surjective, thus so is their composition, and, since $\mathcal{L}(f) \circ \lambda_A = \lambda_B \circ \varphi_f$, it follows that $\mathcal{L}(f)$ is surjective.

By Remark 1, Lemma 4, (ii), and Lemma 7, for all $\alpha, \beta \in \mathcal{K}(A)$, the following hold: $\mathcal{L}(f)(\widehat{\alpha} \wedge \widehat{\beta}) = \mathcal{L}(f)(\lambda_A(\alpha) \wedge \lambda_A(\beta)) = \mathcal{L}(f)(\lambda_A([\alpha, \beta]_A)) = \lambda_B(\varphi_f([\alpha, \beta]_A)) = \lambda_B(f([\alpha, \beta]_A \vee \mathrm{Ker}(f))) = \lambda_B([f(\alpha \vee \mathrm{Ker}(f)), f(\beta \vee \mathrm{Ker}(f))]_B)) = \lambda_B(f(\alpha \vee \mathrm{Ker}(f))) \wedge \lambda_B(f(\beta \vee \mathrm{Ker}(f))) = \lambda_B(\varphi_f(\alpha)) \wedge \lambda_B(\varphi_f(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha)) \wedge \mathcal{L}(f)(\lambda_A(\beta)) = \mathcal{L}(f)(\widehat{\alpha}) \wedge \mathcal{L}(f)(\widehat{\beta})$ and $\mathcal{L}(f)(\widehat{\alpha} \vee \widehat{\beta}) = \mathcal{L}(f)(\lambda_A(\alpha) \vee \lambda_A(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha \vee \beta)) = \lambda_B(\varphi_f(\alpha \vee \beta)) =$

$\lambda_B(f(\alpha\vee\beta\vee\mathrm{Ker}(f))) = \lambda_B(f(\alpha\vee\mathrm{Ker}(f)\vee\beta\vee\mathrm{Ker}(f))) = \lambda_B(f(\alpha\vee\mathrm{Ker}(f)))\vee \lambda_B(f(\beta\vee\mathrm{Ker}(f))) = \lambda_B(\varphi_f(\alpha))\vee\lambda_B(\varphi_f(\beta)) = \mathcal{L}(f)(\lambda_A(\alpha))\vee\mathcal{L}(f)(\lambda_A(\beta)) = \mathcal{L}(f)(\widehat{\alpha})\vee\mathcal{L}(f)(\widehat{\beta})$. Therefore $\mathcal{L}(f)$ is a lattice morphism. □

Remark 18 *Clearly, if C is an algebra from $\mathcal{C}$ and $g : B \to C$ is a surjective morphism in $\mathcal{C}$, then $\mathcal{L}(g\circ f) = \mathcal{L}(g)\circ\mathcal{L}(f)$. Hence we have defined a covariant functor $\mathcal{L}$ from the partial category of $\mathcal{C}$ whose morphisms are exactly the surjective morphisms from $\mathcal{C}$ to the partial category of the category $\mathcal{D}$**01** of bounded distributive lattices whose morphisms are exactly the surjective morphisms from $\mathcal{D}$**01**.*

Open problem 2 *Extend the definition of $\mathcal{L}$ to the whole category $\mathcal{C}$, with the image in $\mathcal{D}$**01**, of course.*

Remark 19 *By Proposition 15, if $\mathcal{C}$ is congruence–distributive, then we may take $\mathcal{L}(f) = \varphi_f\mid_{\mathcal{K}(A)}: \mathcal{K}(A) \to \mathcal{K}(B)$, with $\mathcal{K}(A)$ and $\mathcal{K}(B)$ bounded sublattices of $\mathrm{Con}(A)$ and $\mathrm{Con}(B)$, respectively.*

For any bounded lattice morphism $h : L \to M$, let us denote by $\mathrm{Ker}_{\mathrm{Id}}(h) = h^{-1}(\{0\}) = \{x \in L \mid h(x) = 0\} \in \mathrm{Id}(L)$, so that $L/\mathrm{Ker}_{\mathrm{Id}}(h) \cong h(L)$ by the Main Isomorphism Theorem. Let us prove that the reticulation preserves quotients:

Theorem 7 *For any $\theta \in \mathrm{Con}(A)$, the lattices $\mathcal{L}(A/\theta)$ and $\mathcal{L}(A)/\theta^*$ are isomorphic.*

Proof: Recall that $\theta^* = \lambda_A(\mathcal{K}(A) \cap (\theta]) = \{\widehat{\alpha} \mid \alpha \in \mathcal{K}(A), \alpha \subseteq \theta\} \in \mathrm{Id}(\mathcal{L}(A))$. $p_\theta : A \to A/\theta$ is a surjective morphism in $\mathcal{C}$, so we can apply the construction above:

$$\begin{array}{ccc}
\mathrm{Con}(A) & \xrightarrow{\varphi_{p_\theta}} & \mathrm{Con}(A/\theta) \\
\cup\!| & & \cup\!| \\
\mathcal{K}(A) & \xrightarrow{\varphi_{p_\theta}} & \mathcal{K}(A/\theta) \\
\downarrow{\scriptstyle\lambda_A} & & \downarrow{\scriptstyle\lambda_{A/\theta}} \\
\mathcal{L}(A) & \xrightarrow{\mathcal{L}(p_\theta)} & \mathcal{L}(A/\theta)
\end{array}$$

For all $\alpha \in \mathrm{Con}(A)$, $\varphi_{p_\theta}(\alpha) = p_\theta(\alpha \vee \mathrm{Ker}(p_\theta)) = (\alpha \vee \theta)/\theta$, so, for all $\alpha \in \mathcal{K}(A)$, $\mathcal{L}(p_\theta)(\widehat{\alpha}) = \widehat{(\alpha \vee \theta)/\theta} \in \mathcal{L}(A/\theta)$. Thus, for any $\alpha \in \mathcal{K}(A)$: $\widehat{\alpha} \in \mathrm{Ker}_{\mathrm{Id}}(\mathcal{L}(p_\theta))$ iff $\mathcal{L}(p_\theta)(\widehat{\alpha}) = \widehat{\Delta_{A/\theta}}$ iff $\widehat{(\alpha \vee \theta)/\theta} = \widehat{\theta/\theta}$, that is $\lambda_{A/\theta}((\alpha\vee\theta)/\theta) = \lambda_{A/\theta}(\theta/\theta)$, iff $\rho_{A/\theta}((\alpha\vee\theta)/\theta) = \rho_{A/\theta}(\theta/\theta)$ iff $\rho_A(\alpha\vee\theta)/\theta =$

$\rho_A(\theta)/\theta$ iff $\rho_A(\alpha \vee \theta) = \rho_A(\theta)$ iff $\rho_A(\alpha \vee \theta) \subseteq \rho_A(\theta)$ iff $\alpha \vee \theta \subseteq \rho_A(\theta)$ iff $\alpha \subseteq \rho_A(\theta)$ iff $\widehat{\alpha} \in (\rho_A(\theta))^* = \theta^*$, hence $\mathrm{Ker}_{\mathrm{Id}}(\mathcal{L}(p_\theta)) = \theta^*$; we have applied Lemma 6 and Lemma 14, (i). Proposition 23 ensures us that the lattice morphism $\mathcal{L}(p_\theta)$ is surjective, so, from the Main Isomorphism Theorem, we obtain: $\mathcal{L}(A/\theta) \cong \mathcal{L}(A)/\theta^*$. □

Proposition 24 *The lattices $\mathcal{L}(A)$ and $\mathcal{L}(A/\rho_A(\Delta_A))$ are isomorphic.*

Proof: By Lemma 14, (i), and Theorem 7, $\rho_A(\Delta_A)^* = \Delta_A^*$, hence the lattice $\mathcal{L}(A/\rho_A(\Delta_A))$ is isomorphic to $\mathcal{L}(A)/\rho_A(\Delta_A)^* = \mathcal{L}(A)/\Delta_A^*$, which, in turn, is isomorphic to $\mathcal{L}(A/\Delta_A)$, and thus to $\mathcal{L}(A)$, since the algebras A/Δ_A and A are isomorphic. □

Remark 20 *Propositions 17 and 24 show that the reticulation of any algebra M from a semi–degenerate congruence–modular variety such that $\mathcal{K}(M)$ is closed with respect to the commutator of M and $\nabla_M \in \mathcal{K}(M)$ is isomorphic to the reticulation of a semiprime algebra from the same variety.*

Corollary 3 *$\mathcal{B}(\mathcal{L}(A))$ and $\mathcal{B}(\mathrm{Con}(A/\rho_A(\Delta_A)))$ are isomorphic Boolean algebras.*

Proof: By Proposition 17, Theorem 5 and Proposition 24, $A/\rho_A(\Delta_A)$ is semiprime, thus the Boolean algebra $\mathcal{B}(\mathrm{Con}(A/\rho_A(\Delta_A)))$ is isomorphic to $\mathcal{B}(\mathcal{L}(A/\rho_A(\Delta_A)))$, which in turn is isomorphic to $\mathcal{B}(\mathcal{L}(A))$. □

Recall **Nachbin's Theorem**, stating that, if L is a bounded distributive lattice, then: L is a Boolean algebra iff $\mathrm{Max}_{\mathrm{Id}}(L) = \mathrm{Spec}_{\mathrm{Id}}(L)$ iff $\mathrm{Max}_{\mathrm{Filt}}(L) = \mathrm{Spec}_{\mathrm{Filt}}(L)$.

Theorem 8 *The following are equivalent:*

(i) *A is hyperarchimedean;*

(ii) *$A/\rho_A(\Delta_A)$ is hyperarchimedean;*

(iii) $\mathrm{Max}(A) = \mathrm{Spec}(A)$*;*

(iv) *$\mathcal{L}(A)$ is a Boolean lattice;*

(v) *the lattice $\mathcal{L}(A)$ is isomorphic to $\mathcal{B}(\mathrm{Con}(A))$;*

(vi) *the lattice $\mathcal{L}(A)$ is isomorphic to $\mathcal{B}(\mathrm{Con}(A/\rho_A(\Delta_A)))$.*

Proof: By Nachbin‘s Theorem, Proposition 11 and Corollary 1, (iii) is equivalent to (iv). Trivially, (vi) implies (iv), while the converse holds by Corollary 3.

If A is semiprime, that is $\rho_A(\Delta_A) = \Delta_A$, so that $A/\rho_A(\Delta_A) = A/\Delta_A$ is isomorphic to A, then (i) is equivalent to (ii) and (v) is equivalent to (vi). Now let us drop the condition that A is semiprime. But $A/\rho_A(\Delta_A)$ is semiprime, according to Proposition 17, hence, by the above, (ii) is equivalent to $\mathrm{Max}(A/\rho_A(\Delta_A)) = \mathrm{Spec}(A/\rho_A(\Delta_A))$ and to the fact that $\mathcal{L}(A/\rho_A(\Delta_A))$ is a Boolean lattice, which, in turn, is equivalent to (iv) by Proposition 24. Finally, to prove that (ii) is equivalent to (iii), use the above and the fact that, as shown by Lemma 2, $\mathrm{Max}(A/\rho_A(\Delta_A)) = \mathrm{Spec}(A/\rho_A(\Delta_A))$ iff $\mathrm{Max}(A) \cap [\rho_A(\Delta_A)) = \mathrm{Spec}(A) \cap [\rho_A(\Delta_A))$ iff $\mathrm{Max}(A) = \mathrm{Spec}(A)$, since $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A) \subseteq [\rho_A(\Delta_A))$. $\square$

Note that Theorem 8 extends Nachbin‘s Theorem, as well as Kaplansky‘s characterization for regular rings in the sense of Von Neumann: [28, Theorem 3.6, p. 198].

8 Conclusions

The main contributions of the present paper are:

- the construction of the reticulation of a universal algebra using commutator theory, and the proof for its uniqueness: Theorem 3 from Section 4;

our construction generalizes all cases existing in the present litterature and can be used for other classes of algebras, as well; this construction allows a transfer of properties between these classes of algebras and the variety of bounded distributive lattices, which we shall further illustrate in a sequel of this work;

- the preservation theorems for finite direct products and quotients: Theorem 4 from Section 5 and Theorem 7 in Section 7;
- studying the relation between the Boolean center of the congruence lattice of an algebra and the Boolean center of the reticulation of that algebra: Theorem 5 from Section 6;

these two Boolean algebras turn out to be isomorphic in the case of semiprime algebras, as well as that of algebras with associative commutators; in the sequel, we will use this Boolean isomorphism to study Stone, normal and B–normal algebras and some lifting properties; as a theme for future research, this isomorphism theorem could lead to new results for varieties with Boolean Factor Congruences [4];

- establishing the relation between the annihilators in the reticulation of an algebra and the annihilators of the congruences of that algebra in its congruence lattice: Theorem 6 from Section 6;
- a characterization theorem for hyperarchimedean algebras: Theorem 8 of Section 7;

an interesting case for a further study of the reticulation is that of subtractive varieties, using the ideals of the members of these varieties [3].

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