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(Skew) Filters in Residuated Skew Lattices

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Abstract

In this paper, we show the relationship between (skew) deductive system and (skew) filter in residuated skew lattices. It is shown that if a residuated skew lattice is conormal, then any skew deductive system is a skew filter under a condition and deductive system and skew deductive system are equivalent under some conditions too. It is investigated that in branchwise residuated skew lattice, filter, deductive system and skew deductive system are equivalent. We define some types of prime (skew) filters in residuated skew lattices and show the relationship between prime (skew) filters and residuated skew chains. It is proved that in prelinear residuated skew lattice any proper filter can be extended to a maximal, prime filter of type (I). The notion of the radical of a filter is defined and several characterizations of the radical of a filter are given. We show that in non conormal prelinear residuated skew lattice with element 0, infinitesimal elements are equal to intersection of all the maximal filters.

Keywords: Residuated skew lattice, (skew) deductive system, (maximal, prime) (skew) filter

1 Introduction

Non-commutative lattices have been studied for over sixty-five years. To our knowledge, the first person to engage in their extended study was the physicist P. Jordan who published numerous articles on the subject over a span of thirteen years [7]. Since then papers on the subject have been

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written by various authors from a variety of perspectives [13]. The study of skew lattices began with the 1989 paper of Leech [12]. Skew lattices are a generalization of lattices that operations $\vee, \wedge$ are not commutative. In skew lattices, two different order concepts can be defined, the natural preorder and the natural partial order. For given a skew lattice $(A, \vee, \wedge)$ both algebraic reducts $(A, \vee)$ and $(A, \wedge)$ are bands, that is semigroups whose elements are idempotent. The Green's relation is a congruence relation on the skew lattice and its quotient algebra is a lattice. Unlike lattices, the admissible Hasse diagram representing the order structure of a skew lattice can not determine its algebraic structure. J. Leech defined normal and conormal skew lattices. A skew lattice A is normal, if any downset of A is a lattice and is conormal, if any upset of A is a lattice. Any conormal skew lattice with a bottom element is a lattice and any normal skew lattice with a top element is a lattice too [10].

The notion of ideals of skew lattices was first mentioned in [1]. Two natural concepts of ideal can be derived, respectively, from the two concepts of order that arise in the context of skew lattices. J. Pita Costa called them skew ideals and ideal and expressed notions of (skew) ideals and (skew) filters in skew lattices [15]. The filter theory for logical algebras plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. Various logical algebras have been proposed as a semantic system for non-classical logic system, such as residuated lattices, BL-algebra, MV-algebra. Residuated lattices are very useful algebraic structures because they describe the structure of truth values of fuzzy logic. Filters are tools of extreme importance in studying these logical algebras and the completeness of non-classical logics. Residuated lattices originated in Mathematical Logic without contraction. They have been investigated by Krull [9]. Residuated lattices, introduced by Ward and Dilworth in [17], are a common structure among algebras associated with logical systems. Chajda et.al introduced a non-commutative generalization of the residuated lattice and called it skew residuated lattice [3]. Another non-commutative generalization of the residuated lattice is given by residuum on the skew lattice which is called residuated skew lattice [2, 18]. We have studied residuated skew lattices and our results are a generalization of what is defined in [18]. In the paper [2], A. Borumand Saeid and R. Koohnavard defined residuated skew lattices as non-commutative generalization of residuated lattices. They defined deductive system and skew deductive system in residuated skew lattices and investigated relationships between them and showed that the class of all conormal residuated skew lattices forms a variety too. They showed that the Green's relation is a congruence relation on residuated skew lattice and its quotient algebra is a residuated lattice.

For better understanding of residuated skew lattices, we investigate filters and use of them for classification and obtain residuated skew lattices structural properties. The filter theory in residuated skew lattices plays an important role. From a logic point of view, various filters have natural interpretation as various sets of provable formulas. In this paper, we study the relationship between (skew) deductive system and (skew) filter in residuated skew lattice and show that deductive system and filter are equivalent. On the other hand if a residuated skew lattice is conormal, then any skew deductive system is a skew filter under a condition and deductive system and skew deductive system are equivalent under some conditions too. It is investigated that in branchwise residuated skew lattice, filter, deductive system and skew deductive system are equivalent. We generalize the concept of prime filters in residuated lattices and use it for classification of residuated skew lattices and the relationship between residuated skew chains and residuated skew lattices. We show that F is a prime (skew) filter of type (II) iff its quotient algebra is a residuated skew chain and if A is a prelinear residuated skew lattice, then any proper filter can be extended to a maximal, prime filter of type (I) too. $\vee$-irreducible elements are defined and is shown that $\{1\}$ is $\vee$-irreducible iff $\{1\}$ is a prime (skew) filter of type (I). We introduce the concept of the radical of a filter and investigate some of its properties. We show that in non conormal prelinear residuated skew lattice with 0, infinitesimal elements are equal to intersection of all the maximal filters.

2 Preliminaries

In this section, we review some properties of skew lattices and residuated skew lattices which we need in the sequel.

Definition 2.1 **([1])** *A skew lattice is an algebra* $(A, \vee, \wedge)$ *of type* $(2, 2)$ *such that satisfies in the following identities:*

(1) $(x \vee y) \vee z = x \vee (y \vee z)$ *and* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,

(2) $x \wedge x = x$ *and* $x \vee x = x$,

(3) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ *and* $(x \wedge y) \vee y = y = (x \vee y) \wedge y$,

The identities found in $(1-3)$ *are known as the associative law, the idempotent laws and absorption laws respectively. In view of the associativity* (1)*, we can omit parentheses when no ambiguity arises.*

On a given skew lattice A the natural partial order $\leq$ and natural preorder $\preceq$ respectively are defined by $x \leq y$ iff $x \wedge y = x = y \wedge x$ or dually $x \vee y = y = y \vee x$ and $x \preceq y$ if and only if $y \vee x \vee y = y$ or equivalently $x \wedge y \wedge x = x$. Relation $\mathbb{D}$ is defined by $x \,\mathbb{D}\, y$ iff $x \vee y \vee x = x$ and $y \vee x \vee y = y$ or dually, $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. For any elements x, y of a skew lattice A, $x \,\mathbb{D}\, y$ iff $x \vee y = y \wedge x$ [5].

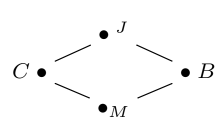
All lattices are skew lattices. $(A, \wedge)$ is a rectangular band, if $\wedge$ is idempotent and associative, $x \wedge y \wedge z = x \wedge z$. At the other extreme, a rectangular skew lattice is a skew lattice A for which both $(A, \wedge)$ and $(A, \vee)$ are rectangular bands whose multiplications dualize each other, $x \wedge y = y \vee x$ [11]. In fact,

Definition 2.2 ([4]) *A skew lattice is rectangular if it satisfies the identity* $x \vee y = y \wedge x$.

Lemma 2.1 ([4]) *The following conditions are equivalent for a skew lattice A (for all $x, y, z \in A$):*

(1) *A is rectangular,*
(2) $x \wedge y \wedge x = x$,
(3) $x \vee (y \wedge x) = y \wedge x$,
(4) $(y \wedge x) \vee y = y \wedge x$,
(5) $x \vee y \vee x = x$,
(6) $(x \vee y) \wedge x = x \vee y$,
(7) $y \wedge (x \vee y) = x \vee y$,
(8) $x \wedge y \wedge z = x \wedge z$,
(9) $x \vee y \vee z = x \vee z$.

The First Decomposition Theorem for skew lattices states: each $\mathbb{D}$-class is a maximal rectangular subalgebra of A and $A/\mathbb{D}$ is the maximal lattice image of A. In brief, every skew lattice is a lattice of rectangular subalgebras. In the below figure, C, B, J, M are maximal rectangular subalgebra.



Clearly $x \succeq y$ in A iff $\mathbb{D}_x \geq \mathbb{D}_y$ in the lattice $A/\mathbb{D}$ where $\mathbb{D}_x$ and $\mathbb{D}_y$ are the $\mathbb{D}$-classes of x and y, respectively. Given $a \in C$ and $b \in B$ for $\mathbb{D}$-classes C and B, $a \vee b$ just lie in their join $\mathbb{D}$-class J; similarly $a \wedge b$ must lie in their meet $\mathbb{D}$-class M [8].

We say that x is $\mathbb{L}$-related to y (denoted $x \,\mathbb{L}\, y$) if $x \wedge y = x$ and $y \wedge x = y$, or dually, $x \vee y = y$ and $y \vee x = x$. x and y are $\mathbb{R}$-related ($x \,\mathbb{R}\, y$) if $x \wedge y = y$ and $y \wedge x = x$, or dually, $x \vee y = x$ and $y \vee x = y$ [5]. The equivalences $\mathbb{L}$ and $\mathbb{R}$, induced by $\preceq$, are again congruences [16]. $\mathbb{L}$, $\mathbb{R}$ and $\mathbb{D}$ are congruences on any skew lattice, with $\mathbb{L} \vee \mathbb{R} = \mathbb{L} \circ \mathbb{R} = \mathbb{R} \circ \mathbb{L} = \mathbb{D}$ and $\mathbb{L} \cap \mathbb{R} = \triangle$, the identity equivalence. Their congruence classes (called $\mathbb{R}$-classes, $\mathbb{L}$-classes or $\mathbb{D}$-classes) are all rectangular subalgebras [8].

Definition 2.3 ([16]) *A skew lattice is left-handed (right-handed) if* $\mathbb{L} = \mathbb{D}$ $(\mathbb{R} = \mathbb{D})$.

Lemma 2.2 ([16]) *The next conditions are equivalent for a skew lattice A:*

(1) *A is left-handed,*
(2) *for all $x, y \in A$, $x \,\mathbb{D}\, y$ implies $x \wedge y = x$,*
(3) *for all $x, y \in A$, $x \wedge y \wedge x = x \wedge y$.*

Definition 2.4 ([10]) *Skew lattice A is normal if $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$ and is conormal if $x \vee y \vee z \vee w = x \vee z \vee y \vee w$ for all $x, y, z, w \in A$.*

Proposition 2.1 ([10]) *A skew lattice A is normal iff each sub skew lattice $(\downarrow x)$ is a sub lattice of A. Dually, A is conormal iff each sub skew lattice $(\uparrow x)$ is a sub lattice of A $((\uparrow x) = \{y \in A | y \geq x\}, (\downarrow x) = \{y \in A | y \leq x\})$.*

A skew lattice is distributive if it satisfies $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$ [11].

A skew chain is a skew lattice where $A/\mathbb{D}$ is a chain i.e. for all $x, y \in A$, $x \preceq y$ or $y \preceq x$ [6].

Definition 2.5 ([2]) *A residuated skew lattice is a nonempty set A with binary operations $\vee, \wedge, \odot$ and hyperoperation $\rightarrow$ and constant element 1 that satisfying the following:*

(1) *$(A, \vee, \wedge, 1)$ is a skew lattice with top 1 (for all $x \in A$, $x \leq 1$),*
(2) *$(A, \odot, 1)$ is a commutative monoid,*
(3) *$\odot$ and $\rightarrow$ form an adjoint pair, i.e. $z \preceq x \rightarrow y$ iff $x \odot z \preceq y$, for all $x, y, z \in A$.*

The relations between the pair of operations $\odot$ and $\rightarrow$ expressed by (3), is a special case of the law of residuation and for every $x, y \in A, x \rightarrow y = sup\{z \in A | x \odot z \preceq y\}$. Supremum of a set in a pre-ordered set is not a unique element, $x \to y$ may be a $\mathbb{D}$-class (for example $x \to y = \mathbb{D}_a = \{a, b\}$ that $a\,\mathbb{D}\,b$). Two $\mathbb{D}$-classes have $\mathbb{D}$-relationship when all of their members have $\mathbb{D}$-relationship with each other. Relation $\preceq$ between two $\mathbb{D}$-classes is defined member to member (i.e. $B \preceq C$ iff $\forall c \in C, \forall b \in B, b \preceq c$) [2].

Lemma 2.3 **([2])** *If A is a residuated skew lattice and $x, y, z \in A$, then*

(1) $1 \to x = \mathbb{D}_x$ *and* $x \to x = 1$,
(2) $x \odot y \preceq x, y$ *hence* $x \odot y \preceq x \wedge y, y \wedge x$, $y \preceq x \to y$,
(3) $x \odot y \preceq x \to y$,
(4) $x \preceq y$ *iff* $x \to y = 1$ *and* $x\,\mathbb{D}\,y$ *iff* $x \to y = y \to x = 1$,
(5) $x \to 1 = 1$,
(6) $x \odot (x \to y) \preceq y$, $x \preceq (x \to y) \to y$ *and* $((x \to y) \to y) \to y)\,\mathbb{D}\,(x \to y)$,
(7) $x \to y \preceq x \odot z \to y \odot z$,
(8) $x \preceq y$ *implies* $x \odot z \preceq y \odot z$,
(9) $x \to y \preceq (z \to x) \to (z \to y)$,
(10) $x \to y \preceq (y \to z) \to (x \to z)$,
(11) $x \preceq y$ *implies* $z \to x \preceq z \to y$ *and* $y \to z \preceq x \to z$,
(12) $x \odot (y \to z) \preceq y \to (x \odot z) \preceq x \odot y \to x \odot z$,
(13) $(x \to (y \to z))\,\mathbb{D}\,(x \odot y \to z)\,\mathbb{D}\,(y \to (x \to z))$,
(14) $x_1 \to y_1 \preceq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$,
(15) $x \vee y \preceq (x \to y) \to y \wedge (y \to x) \to x \wedge (x \to y) \to y$,
(16) $x \odot (x \to y) \preceq (x \wedge y)$, $x \odot (x \to y) \preceq (y \wedge x)$.

Corollary 2.1 **([2])** *Let A be a residuated skew lattice with 0. For every $x \in A$ and $n \geq 1$ we have $(x^{**})^n \preceq (x^n)^{**}$, $((x^{**})^n = x^{**} \odot \cdots \odot x^{**})$, $(x^* = x \to 0)$.*

Definition 2.6 **([2])** *A branchwise residuated skew lattice is an algebra $A = (A, \vee, \wedge, \odot, \to, 1)$ of type $(2, 2, 2, 2, 0)$ satisfying the following:*

(*i*) $(A, \vee, \wedge, 1)$ *is a distributive skew lattice with top* 1 *(for all $x \in A$, $x \leq 1$),*
(*ii*) $(A, \odot, 1)$ *is a commutative monoid,*
(*iii*) *For any $u \in A$, two operations $\to_u, \odot_u$ can be defined on $(\uparrow u)$ such that $(\uparrow u, \vee, \wedge, \odot_u, \to_u, u, 1)$ is a distributive residuated lattice by top 1 and bottom u $(\uparrow u = \{x \in A | x \geq u\})$,*
(*iv*) $x \to y = (y \vee x \vee y) \to_y y$,
(*v*) $x \odot y\,\mathbb{D}\,x \odot_u y$, *for all* $u \in A$, $x, y \in (\uparrow u)$.

Theorem 2.1 ([2]) *If A is a conormal distributive residuated skew lattice which $x \odot y \ \mathbb{D} \ x \odot_u y$, for every $u \in A$, $x, y \in \ (\uparrow u)$, then branchwise residuated skew lattice and residuated skew lattice are equivalent.*

Definition 2.7 ([2]) *Let A be a residuated skew lattice. A nonempty subset $D \subseteq A$ is called a deductive system (for short ds) of A, if the following conditions are satisfied:*

(i) $1 \in D$,

(ii) *If* $x \in D, x \to y \subseteq D$, *then* $y \in D$.

Definition 2.8 ([2]) *Let A be a residuated skew lattice. A nonempty subset $D \subseteq A$ is called a skew deductive system of A, if the following conditions are satisfied:*

(i) $1 \in D$,

(ii) *If* $x \in D, x \to y \subseteq D$, *then* $y \wedge (x \to y) \wedge y \subseteq D$.

Definition 2.9 ([2]) *A deductive system of a residuated skew lattice A is maximal if it is proper and it is not contained in any other proper deductive system.*

Proposition 2.2 ([2]) *Let $S \subseteq A$ be a nonempty subset of A, $x \in A$ and D, D_1, D_2 be deductive systems. Then*

(1) *If S is a deductive system, then $[S) = S$,*

(2) $[S) = \{y \in A | s_1 \odot \ldots \odot s_n \preceq y,$ *for some* $n \geq 1$ *and* $s_1, \ldots, s_n \in S\}$. *In particular,* $[x) = \{y \in A | y \succeq x^n,$ *for some* $n \geq 1\}$,

(3) $D(x) = \{y \in A | y \succeq d \odot x^n,$ *which* $d \in D$ *and* $n \geq 1\}$,

(4) $[D_1 \cup D_2) = \{y \in A | y \succeq d_1 \odot d_2$ *for some* $d_1 \in D_1$ *and* $d_2 \in D_2\}$.

Corollary 2.2 ([2]) *If M is a proper deductive system of residuated skew lattice A with 0, then the following are equivalent:*

(i) *M is a maximal deductive system,*

(ii) *For any* $x \in A, x \notin M$ *iff* $(x^n)^* \subseteq M$, *for some* $n \geq 1$.

3 Prime (Skew) Filters

In this section, we define (skew) filter and show that there is a relationship between it and (skew) deductive system. For classification of residuated skew lattices and the relationship between residuated skew chains and filters, we define prime filters too. The prime filters in residuated skew lattices are a generalization of the concept of the prime filters in residuated lattices. From here to the end of the paper, let A be a residuated skew lattice, unless otherwise stated.

Definition 3.1 *A nonempty subset* $F \subseteq A$ *is called a filter of* A*, if the following conditions are satisfied:*

(*i*) *If* $x \in F$ *and* $x \preceq y$*, then* $y \in F$*,*
(*ii*) *If* $x, y \in F$*, then* $x \odot y \in F$*.*

In the following diagrams, the natural partial order is indicated by full line and the congruence $\mathbb{D}$ is indicated by dashed line.

Example 3.1 *Let* $A = \{0', 0, m, a, b, p, n, c, d, 1\}$ *be a skew lattice such that* $0', 0 < m < a, b < p < n < c, d < 1$ *and* $0 \,\mathbb{D}\, 0'$*,* $a \,\mathbb{D}\, b$*,* $\mathbb{D}_a = \{a, b\}$*,* $\mathbb{D}_0 = \{0, 0'\}$*.* $A = (A, \vee, \wedge, \odot, \rightarrow, \mathbb{D}_0, 1)$ *is a residuated skew lattice with the following operations:*

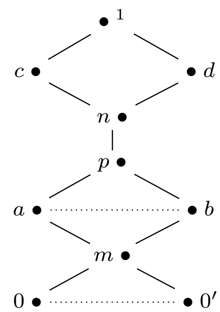
$\rightarrow$	0	$0'$	m	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1	1	1
$0'$	1	1	1	1	1	1	1	1	1	1
m	$\mathbb{D}_0$	$\mathbb{D}_0$	1	1	1	1	1	1	1	1
a	$\mathbb{D}_0$	$\mathbb{D}_0$	$\mathbb{D}_a$	1	1	1	1	1	1	1
b	$\mathbb{D}_0$	$\mathbb{D}_0$	$\mathbb{D}_a$	1	1	1	1	1	1	1
p	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	1	1	1	1	1
n	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	1	1	1	1
c	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	d	1	d	1
d	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	c	c	1	1
1	$\mathbb{D}_0$	$\mathbb{D}_0$	m	$\mathbb{D}_a$	$\mathbb{D}_a$	p	n	c	d	1

$\odot$	0	$0'$	m	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0	0	0
$0'$	0	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$
m	0	$0'$	m	m	m	m	m	m	m	m
a	0	$0'$	m	m	m	a	a	a	a	a
b	0	$0'$	m	m	m	b	b	b	b	b
p	0	$0'$	m	a	b	p	p	p	p	p
n	0	$0'$	m	a	b	p	n	n	n	n
c	0	$0'$	m	a	b	p	n	c	n	c
d	0	$0'$	m	a	b	p	n	n	d	d
1	0	$0'$	m	a	b	p	n	c	d	1

$\vee$	0	$0'$	m	a	b	p	n	c	d	1
0	0	$0'$	m	a	b	p	n	c	d	1
$0'$	0	$0'$	m	a	b	p	n	c	d	1
m	m	m	m	a	b	p	n	c	d	1
a	a	a	a	a	b	p	n	c	d	1
b	b	b	b	a	b	p	n	c	d	1
p	p	p	p	p	p	p	n	c	d	1
n	n	n	n	n	n	n	n	c	d	1
c	c	c	c	c	c	c	c	c	1	1
d	d	d	d	d	d	d	d	1	d	1
1	1	1	1	1	1	1	1	1	1	1

$\wedge$	0	$0'$	m	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0	0	0
$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$0'$
m	0	$0'$	m	m	m	m	m	m	m	m
a	0	$0'$	m	a	a	a	a	a	a	a
b	0	$0'$	m	b	b	b	b	b	b	b
p	0	$0'$	m	a	b	p	p	p	p	p
n	0	$0'$	m	a	b	p	n	n	n	n
c	0	$0'$	m	a	b	p	n	c	n	c
d	0	$0'$	m	a	b	p	n	n	d	d
1	0	$0'$	m	a	b	p	n	c	d	1

$F_1 = \{n, c, d, 1\}$, $F_2 = \{m, a, b, p, n, c, d, 1\}$, $F_3 = \{d, 1\}$ *and* $F_4 = \{p, n, c, d, 1\}$ *are filters of* A*.*



Definition 3.2 *A nonempty subset $F \subseteq A$ is called a skew filter of A, if the following conditions are satisfied:*

(i) *If $x \in F$ and $x \leq y$, then $y \in F$,*

(ii) *If $x, y \in F$, then $x \odot y \in F$.*

Since $x \leq y$ implies $x \preceq y$, therefore any filter of A is a skew filter.

Example 3.2 *In Example 3.1, $F = \{0', m, a, b, p, n, c, d, 1\}$ is a skew filter but is not a filter, since $0' \preceq 0$ and $0' \in F$ but $0 \notin F$.*

Proposition 3.1 *A nonempty subset $F \subseteq A$ is a filter of A iff F is a deductive system of A.*

Proof: *Let F be a deductive system of A, $x \preceq y$ and $x \in F$. So $x \to y = 1 \in F$ therefore $y \in F$. Now, let $x, y \in F$. We must show that $x \odot y \in F$. Since $x \odot y \to x \odot y = 1 \in F$, $x, y \in F$ and $x \odot y \to x \odot y \,\mathbb{D}\, x \to (y \to x \odot y)$ therefore $x \to (y \to x \odot y) \subseteq F$. Thus we deduce $x \odot y \in F$.*

Conversely, let F be a filter of A, $x \preceq 1$ and $x \in F$. By assumption, $1 \in F$. Now, let $x \in F, x \to y \subseteq F$. Thus $x \odot (x \to y) \in F$ and since $x \odot (x \to y) \preceq y$ we get that $y \in F$. □

Proposition 3.2 *A nonempty subset $F \subseteq A$ is a filter of A iff $x \in F, x \to y \cap F \neq \emptyset$ imply $y \in F$.*

Proof: *Let F be a filter and $x \in F, x \to y \cap F \neq \emptyset$. We implies $x \in F, x \to y \subseteq F$. Therefore $y \in F$.*

Conversely, if $x \in F$, $x \to y \subseteq F$, then $x \in F$, $x \to y \cap F \neq \emptyset$. Therefore $y \in F$. □

Any branch of a conormal residuated skew lattice is a residuated lattice. In the following, we want to study the relationship between deductive system, skew deductive system and skew filter in the conormal residuated skew lattices and branchwise residuated skew lattices.

Proposition 3.3 *Let A be a conormal residuated skew lattice such that $x \to y = y \vee x \vee y \to_y y$ (for all $x, y \in A$). If $F \subseteq A$ is a skew deductive system, then F is a deductive system of A.*

Proof: *Let $x \in F, x \to y \subseteq F$. We must show that $y \in F$. Since F is a skew deductive system, then $y \wedge (x \to y) \wedge y \in F$. On the other hand*

$$y = y \wedge (y \vee x \vee y \to_y y) \wedge y = y \wedge (x \to y) \wedge y$$

therefore $y \in F$. □

Corollary 3.1 *Let A be a conormal residuated skew lattice such that $x \to y = y \vee x \vee y \to_y y$ (for all $x, y \in A$). Then $F \subseteq A$ is a skew deductive system iff F is a deductive system of A.*

Proof: *By Proposition 3.3 and Lemma 4.1 of [2], it is clear.* □

Theorem 3.1 *Let A be a conormal residuated skew lattice such that $x \to y = y \vee x \vee y \to_y y$ (for all $x, y \in A$). If $F \subseteq A$ is a skew deductive system, then F is a skew filter of A.*

Proof: *Let F be a skew deductive system, $x \leq y$ and $x \in F$. Therefore $x \to y = 1 \in F$ and $y \wedge (x \to y) \wedge y = y \wedge 1 \wedge y = y \in F$. Now, let $x, y \in F$. Since $1 = (x \odot y) \to (x \odot y) \in F$, then $(y \to x \odot y) \wedge (x \to (y \to x \odot y)) \wedge (y \to x \odot y) \subseteq F$. Therefore $y \to x \odot y \subseteq F$, this implies $(x \odot y) \wedge (y \to x \odot y) \wedge (x \odot y) \in F$. On the other hand we have $y \to x \odot y = ((x \odot y) \vee y \vee (x \odot y)) \to_{(x \odot y)} (x \odot y) \in \uparrow (x \odot y)$ i.e. $y \to x \odot y \in \uparrow (x \odot y)$. Therefore $x \odot y = (x \odot y) \wedge (y \to x \odot y) \wedge (x \odot y) \in F$.* □

Corollary 3.2 *Let A be a branchwise residuated skew lattice and $F \subseteq A$ be a nonempty subset of A. Then*

(*i*) *if F is a skew deductive system of A, then F is a skew filter,*
(*ii*) *F is a deductive system of A iff F is a skew deductive system.*

Any branch in branchwise residuated skew lattices is a residuated lattice, therefore in any branch the relation $\preceq$ coincides the relation $\leq$. Now by above corollary and since any branchwise residuated skew lattice is a residuated skew lattice and Proposition 3.1, we have filter, deductive system and skew deductive system are equivalent in branchwise residuated skew lattices.

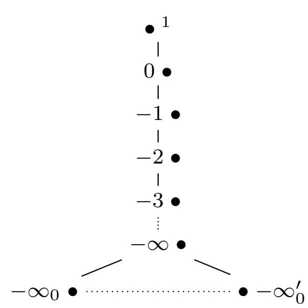
In Example 3.1, $F = \{0', m, a, b, p, n, c, d, 1\}$ is a skew filter, but is not a skew deductive system, since $0' \in F, 0' \to 0 = 1 \in F$ but $0 \wedge (0' \to 0) \wedge 0 = 0 \notin F$.

Converse of Corollary 3.2 (i) does not hold, because:

Example 3.3 *Let $A = \{\mathbb{D}_0, -\infty, ... -3, -2, -1, 0, 1\}$ be a skew lattice such that $\mathbb{D}_0 = \{-\infty_0, -\infty_0'\}$, $-\infty_0 \mathbb{D} -\infty_0'$ and $-\infty_0', -\infty_0 < -\infty < ... < -1 < 0 < 1$. $A = (A, \vee, \wedge, \odot, \rightarrow, 1)$ is an infinitely branchwise residuated skew lattice with the following operations: $-\infty_0 \wedge -\infty_0' = -\infty_0$, $-\infty_0' \wedge -\infty_0 = -\infty_0'$, $-\infty_0 \vee -\infty_0' = -\infty_0'$, $-\infty_0' \vee -\infty_0 = -\infty_0$. Also for all $x, y \in A$, if $x \leq y$, then $x \wedge y = x$ and $x \vee y = y$.*

$\rightarrow$	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	0	1
$-\infty_0$	1	1	1	...	1	1	1	1	1
$-\infty_0'$	1	1	1	...	1	1	1	1	1
$-\infty$	$-\infty_0$	$-\infty_0'$	1	$\cdots$	1	1	1	1	1
$\vdots$	$\vdots$	...	...	...	...	...	...		
-3	$-\infty_0$	$-\infty_0'$	$-\infty$	...	1	1	1	1	1
-2	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	1	1	1	1
-1	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	1	1	1
0	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	1	1
1	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	0	1

$\odot$	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	0	1
$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	...	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$
$-\infty_0'$	$-\infty_0$	$-\infty_0'$	$-\infty_0'$	...	$-\infty_0'$	$-\infty_0'$	$-\infty_0'$	$-\infty_0'$	$-\infty_0'$
$-\infty$	$-\infty_0$	$-\infty_0'$	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	...	...	...	...	...	...		
-3	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-3	-3	-3	-3
-2	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-2	-2	-2
-1	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	-1	-1
0	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	0	0
1	$-\infty_0$	$-\infty_0'$	$-\infty$	...	-3	-2	-1	0	1



$F = \{-\infty_0, -\infty, \ldots, -3, -2, -1, 0, 1\}$ is a skew filter but is not a skew deductive system. Because $-\infty_0 \in F, -\infty_0 \rightarrow -\infty_0' = 1 \in F$ but $-\infty_0' \wedge (-\infty_0 \rightarrow -\infty_0') \wedge -\infty_0' = -\infty_0' \notin F$.

Definition 3.3 *A (skew) filter F of A is called prime (skew) filter of*

(i) *type (I)* ($PF1$), *if $x \vee y \vee x \in F$, then $x \in F$ or $y \in F$, for all $x, y \in A$,*
(ii) *type (II)* ($PF2$), *if $x \to y \subseteq F$ or $y \to x \subseteq F$, for all $x, y \in A$,*
(iii) *type (III)* ($PF3$), *if $(x \to y) \vee (y \to x) \vee (x \to y) \subseteq F$, for all $x, y \in A$.*

Residuated skew lattice A is called prelinear, if $(x \to y) \vee (y \to x) \vee (x \to y) = 1$, for all $x, y \in A$. In Example 3.1, A is prelinear.

Proposition 3.4 (i) *Any prime filter of type (II) is a prime filter of type (I),*
(ii) *If A is prelinear and F is a prime (skew) filter of type (I), then F is a prime (skew) filter of type (II),*
(iii) *Any prime filter of type (II) is a prime filter of type (III),*
(iv) *If F is a prime (skew) filter of type (I) and a prime (skew) filter of type (III) of A, then F is a prime (skew) filter of type (II).*

Proof:

(i) *Let $x \vee y \vee x \in F$ and without loss of generality let A be a left-handed residuated skew lattice. In left-handed residuated skew lattice we have $x \vee y \vee x = y \vee x$ and $y \vee x \preceq (y \to x) \to x \wedge (x \to y) \to y$, we deduce that $(y \to x) \to x, (x \to y) \to y \subseteq F$ and according to the assumption $x \in F$ or $y \in F$. (If A be a right-handed residuated skew lattice, then it is proven similarly).*
(ii) *It is clear.*
(iii) *Let F be a filter. Since $x \to y, y \to x \preceq (x \to y) \vee (y \to x) \vee (x \to y), (y \to x) \vee (x \to y) \vee (y \to x)$, it is clear.*
(iv) *Since F is a prime (skew) filter of type (III), then $(x \to y) \vee (y \to x) \vee (x \to y) \in F$ which implies $(x \to y) \subseteq F$ or $(y \to x) \subseteq F$ since F is a prime (skew) filter of type (I).* □

Remark 3.1 *Let A be prelinear. F is a prime filter of type (I) iff F is a prime filter of type (II).*

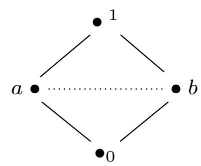
Example 3.4 *Let $A = \{0, a, b, 1\}$ be a skew lattice such that $0 < a, b < 1$, $a \,\mathbb{D}\, b$ and $\mathbb{D}_a = \{a, b\}$. $A = (A, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated skew lattice (residuated skew chain) with the following operations:*

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	$\mathbb{D}_a$	$\mathbb{D}_a$	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	a	b	1
1	1	1	1	1

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
1	0	a	b	1

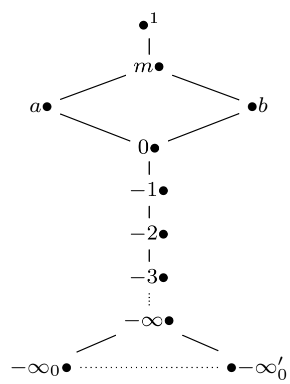


$F = \{b, 1\}$ is a prime skew filter of types (I), (II) and (III), but is not a prime filter. In Example 3.1, $F_1 = \{n, c, d, 1\}$ is a prime filter of types (I), (II) and (III) and $F_2 = \{1\}$ is a prime filter of type (III) but is not of types (I) and (II), because $c \vee d \vee c = 1 \in F_2$ but $c, d \notin F_2$ and $c \to d = d \notin F_2, d \to c = c \notin F_2$.

Example 3.5 *Let $A = \{-\infty_0, -\infty_0', -\infty, ... -3, -2, -1, 0, a, b, m, 1\}$ be a skew lattice such that $\mathbb{D}_0 = \{-\infty_0, -\infty_0'\}$, $-\infty_0 \, \mathbb{D} - \infty_0'$ and $-\infty_0, -\infty_0' < -\infty < ... < -1 < 0 < a < b < m < 1$. $A = (A, \vee, \wedge, \odot, \to, 1)$ is an infinite residuated skew lattice with the following operations: $-\infty_0 \wedge -\infty_0' = -\infty_0$, $-\infty_0' \wedge -\infty_0 = -\infty_0'$, $-\infty_0 \vee -\infty_0' = -\infty_0', -\infty_0' \vee -\infty_0 = -\infty_0$ and $a \wedge b = 0 = b \wedge a$, $a \vee b = m = b \vee a$. If $x \leq y$, then $x \wedge y = x, x \vee y = y$, for all $x, y \in A$.*

$\to$	$-\infty_0$	$-\infty_0'$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	m	1
$-\infty_0$	1	1	1	$\cdots$	1	1	1	1	1	1	1	1
$-\infty_0'$	1	1	1	$\cdots$	1	1	1	1	1	1	1	1
$-\infty$	$\mathbb{D}_0$	$\mathbb{D}_0$	1	$\cdots$	1	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$					
-3	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	1	1	1	1	1	1	1	1
-2	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	1	1	1	1	1	1	1
-1	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	-2	1	1	1	1	1	1
0	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	-2	-1	1	1	1	1	1
a	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	-2	-1	a	1	b	1	1
b	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	-2	-1	b	a	1	1	1
m	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1	1
1	$\mathbb{D}_0$	$\mathbb{D}_0$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	m	1

$\odot$	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	m	1
$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$\cdots$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$
$-\infty'_0$	$-\infty_0$	$-\infty'_0$	$-\infty'_0$	$\cdots$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$	$-\infty'_0$
$-\infty$	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$					
-3	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-3	-3	-3	-3	-3	-3	-3
-2	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-2	-2	-2	-2	-2	-2
-1	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	-1	-1	-1	-1	-1
0	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	0	0	0	0	0
a	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	0	a	0	a	a
b	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	0	0	b	b	b
m	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	m	m
1	$-\infty_0$	$-\infty'_0$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	m	1



In this example, $F = \{1\}$ is a prime filter of type (I) but is not of types (II) and (III), because $(a \to b) = b \notin F, (b \to a) = a \notin F$ and $(a \to b) \vee (b \to a) \vee (a \to b) = m \notin F$.

Proposition 3.5 (i) *Any prime filter of A is a prime skew filter,*

(ii) *Suppose F and G are (skew) filters of A and $F \subseteq G$, then*

(1) *if F is a prime (skew) filter of type (II), then G is prime (skew) filter of type (II),*

(2) *if F is a prime (skew) filter of type (III), then G is prime (skew) filter of type (III).*

Proof:

(i) *Since $\leq$ implies $\preceq$.*

(ii) (1) *By assumption, $x \to y \subseteq F \subseteq G$ or $y \to x \subseteq F \subseteq G$, for all $x, y \in A$.*

(2) *By assumption, $(x \to y) \vee (y \to x) \vee (x \to y) \subseteq F \subseteq G$, for all $x, y \in A$.*

□

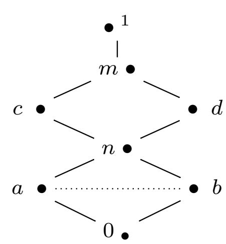
Example 3.6 *Let* $A = \{0, a, b, n, c, d, m, 1\}$ *be such that* $0 < a, b < n < c, d < m < 1$*,* $a \mathbb{D} b$ *and* $\mathbb{D}_a = \{a, b\}$*.* $A = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is a residuated skew lattice with* 0*, with the following operations:*

$\rightarrow$	0	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1
a	$\mathbb{D}_a$	1	1	1	1	1	1	1
b	$\mathbb{D}_a$	1	1	1	1	1	1	1
n	0	$\mathbb{D}_a$	$\mathbb{D}_a$	1	1	1	1	1
c	0	$\mathbb{D}_a$	$\mathbb{D}_a$	d	1	d	1	1
d	0	$\mathbb{D}_a$	$\mathbb{D}_a$	c	c	1	1	1
m	0	$\mathbb{D}_a$	$\mathbb{D}_a$	n	c	d	1	1
1	0	$\mathbb{D}_a$	$\mathbb{D}_a$	n	c	d	m	1

$\odot$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	0	0	a	a	a	a	a
b	0	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	c	n	c	c
d	0	a	b	n	n	d	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1

$\vee$	0	a	b	n	c	d	m	1
0	0	a	b	n	c	d	m	1
a	a	a	b	n	c	d	m	1
b	b	a	b	n	c	d	m	1
n	n	n	n	n	c	d	m	1
c	c	c	c	c	c	m	m	1
d	d	d	d	d	m	d	m	1
m	m	m	m	m	m	m	m	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a
b	0	b	b	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	c	n	c	c
d	0	a	b	n	n	d	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1



$F = \{1\}, G = \{m, 1\}$ are filters of A and F is a prime filter of type (I) but G is not a prime filter of type (I), because $c \vee d \vee c = m \in G$ but $c, d \notin G$ i.e. extension property for prime filter of type (I) does not hold.

The following definitions are generalization of idempotent and $\vee$-irreducible elements in residuated lattices:

Definition 3.4 *$x \in A$ is called an idempotent element, if $x^2 \mathbb{D} x$.*

Example 3.7 *In Example 3.6, $\{0, n, c, d, m, 1\}$ are idempotent elements of A.*

Definition 3.5 *An element $a \in A$ is called $\vee$-irreducible, if $a \,\mathbb{D}\, x \vee y$, then $a \,\mathbb{D}\, x$ or $a \,\mathbb{D}\, y$ (or both).*

Example 3.8 *In Example 3.6, $\{a, b, 1\}$ are $\vee$-irreducible elements.*

If $a \in A$ is an idempotent element, then it is not difficult to see that $F_a = \{x \in A | a \preceq x\}$ is a filter of A.

In following, we investigate the relationship among residuated skew chains, prime (skew) filters and $\vee$-irreducible elements:

Proposition 3.6 (i) *If a is an idempotent and $\vee$-irreducible element of A, then F_a is a prime filter of type (I),*

(ii) *$\{1\}$ is a prime (skew) filter of type (II) iff A is a residuated skew chain iff any filter of A is a prime (skew) filter of type (II),*

(iii) *$\{1\}$ is a prime (skew) filter of type (III) iff A is prelinear iff any (skew) filter of A is a prime (skew) filter of type (III),*

(iv) *$\{1\}$ is a prime (skew) filter of type (I) of A iff $\{1\}$ is $\vee$-irreducible,*

(v) *If any prime filter of type (I) of A is a prime filter of type (II) and $\{1\}$ is $\vee$-irreducible, then A is prelinear.*

Proof:

(i) *Suppose $x \vee y \in F_a$. Then $a \preceq x \vee y$ and therefore $a \,\mathbb{D}\, a \odot a \preceq a \odot (x \vee y) \,\mathbb{D}\, (a \odot x) \vee (a \odot y) \preceq a$. So $a \,\mathbb{D}\, (a \odot x) \vee (a \odot y)$, which implies $a \,\mathbb{D}\, a \odot x$ or $a \,\mathbb{D}\, a \odot y$. So $a \preceq x$ or $a \preceq y$, which means exactly that $x \in F_a$ or $y \in F_a$.*

(ii) *Since $\{1\}$ is a prime (skew) filter of type (II) iff $x \to y = 1$ or $y \to x = 1$ iff $x \preceq y$ or $y \preceq x$ iff A is a residuated skew chain and by Proposition 3.5, it is clear.*

(iii) *is clear.*

(iv) *is clear.*

(v) *By assumption, $(iii), (iv)$ and Proposition 3.4 (iii), we imply that A is prelinear.* □

If F is a filter of A, then define $A/F = \{[x] | x \in A\}$ and $[x] = x/F = \{y \in A | x \to y, y \to x \subseteq F\}$. $\vee, \wedge, \odot, \to$ are defined on A/F as follows: $[x] \vee [y] = [x \vee y]$, $[x] \wedge [y] = [x \wedge y]$, $[x] \odot [y] = [x \odot y]$ and $[x] \to [y] = [x \to y]$. For relation $\preceq$ on A/F, we define $[x] \preceq [y]$ iff $x \to y \subseteq F$, $x, y \in A$.

Proposition 3.7 *If F is a filter of A, then A/F is a residuated skew lattice.*

Proof: *According to associative, idempotent and absorption laws in A, it is clear that $\vee, \wedge$ in A/F are associative, idempotent and absorption. We* *must show that* $x/F \odot y/F \preceq z/F$ *iff* $x/F \preceq y/F \to z/F$ *and* $\vee, \wedge$ *in* A/F *are not commutative. We have* $x/F \odot y/F \preceq z/F$ *iff* $x \odot y \to z \subseteq F$ *iff* $x \to (y \to z) \subseteq F$ *iff* $x/F \preceq (y \to z)/F$ *iff* $x/F \preceq y/F \to z/F$. *Since* $x \vee y \,\mathbb{D}\, y \vee x$, *then* $x \vee y \to y \vee x = 1 \in F, y \vee x \to x \vee y = 1 \in F$ *therefore* $(x \vee y)/F \preceq (y \vee x)/F$ *and* $(y \vee x)/F \preceq (x \vee y)/F$ *so* $x/F \vee y/F \,\mathbb{D}\, y/F \vee x/F$. □

Theorem 3.2 (*i*) *If* F *is a (skew) filter of* A, *then* F *is a prime (skew) filter of type (II) iff* A/F *is a residuated skew chain,*

(*ii*) A/F *is prelinear iff* F *is a prime (skew) filter of type (III).*

Proof:

(*i*) F *is a prime (skew) filter of type (II) iff* $x \to y \subseteq F$ *or* $y \to x \subseteq F$ *iff* $x/F \preceq y/F$ *or* $y/F \preceq x/F$ *iff* A/F *is a residuated skew chain.*

(*ii*) F *is a prime (skew) filter of type (III) iff* $(x \to y) \vee (y \to x) \vee (x \to y) \in F$ *iff* $((x \to y) \vee (y \to x) \vee (x \to y))/F = 1/F$ *iff* A/F *is prelinear.* □

Proposition 3.8 *If* F *is a filter of* A *and* $x \notin F$, *then*

(*i*) *there exists a prime filter* P *of type (I) such that* $F \subseteq P$ *and* $x \notin P$,

(*ii*) F *is the intersection of those prime filters of type (I) which contains* F.

Proof:

(*i*) *We put* $\mathfrak{F} = \{F' \subseteq A | F' \text{ is a } filter, F \subseteq F' \text{ and } x \notin F'\}$. *Clearly* $\mathfrak{F} \neq \emptyset$. *Since any totally ordered subset has a top bound, then by Zorn Lemma,* $\mathfrak{F}$ *has a maximal element as* P. *We show that* P *is a prime filter of type (I). Let* P *be not prime of type (I). Therefore exist* $y, z \in A$ *such that* $z \vee y \vee z \in P$ *but* $y, z \notin P$. *On the other hand since* $P \subset P(y), P(z)$ *and* P *is maximal, then* $P(y), P(z) \notin \mathfrak{F}$. *Therefore* $x \in P(y), P(z)$. *So there exist* $f, g \in P$ *and* $m, n \geq 1$ *such that* $x \succeq f \odot y^m$ *and* $x \succeq g \odot z^n$ *that imply* $x \succeq (f \odot y^m) \vee (g \odot z^n) \succeq (f \vee g) \odot (y^m \vee g) \odot (f \vee z^n) \odot (y^m \vee z^n) \succeq (f \vee g) \odot (y^m \vee g) \odot (f \vee z^n) \odot (y \vee z)^{mn}$ *and since* $z \vee y \vee z \,\mathbb{D}\, y \vee z$, *then* $x \in P$, *that is a contradiction.*

(*ii*) *Let* F_i *(for all* $i \in I$*) be a prime filter of type (I) such that* $F \subseteq F_i$. *Therefore* $F \subseteq \bigcap_{i \in I} F_i$.

Conversely, let $x \in \bigcap_{i \in I} F_i$ *and* $x \notin F$. *By* (*i*), *there exists prime filter* P *of type (I) such that* $F \subseteq P, x \notin P$ *which implies* $x \notin \bigcap_{i \in I} F_i$, *which is a contradiction.* □

Corollary 3.3 *Any proper filter F of A can be extended to a prime filter of type (I).*

Proof: *We put $x = 0$ in Proposition 3.8 (i).* □

Proposition 3.9 *If P is a prime filter of type (II), then the set*

$$\mathfrak{F} = \{F | P \subseteq F, F \text{ is a proper filter}\}$$

is linearly ordered with respect to set-theoretical inclusion.

Proof: *Let $F, G \in \mathfrak{F}$. Assume $G \nsubseteq F, F \nsubseteq G$. Then there are $x, y \in A$ such that $x \in F, x \notin G$ and $y \in G, y \notin F$. Since P is prime of type (II), either $x \to y \subseteq P$ or $y \to x \subseteq P$. If $x \to y \subseteq P \subseteq F$, then $y \in F$, which is a contradiction. If $y \to x \subseteq P \subseteq G$, then $x \in G$, which is a contradiction. Thus $G \subseteq F$ or $F \subseteq G$.* □

Proposition 3.10 *If A is prelinear, then any proper filter can be extended to a maximal, prime filter of type (I).*

Proof: *Let F be a proper filter. By Corollary 3.3, F can be extended to a prime filter of type (I) E, by Remark 3.1, E is a prime filter of type (II), and by Proposition 3.9, the set $\mathfrak{F} = \{G | E \subseteq G,\ G \text{ is a proper filter}\}$ is linearly ordered. Define $M = \bigcup_{G \in \mathfrak{F}} G$. Then $1 \in M$ and if $x, x \to y \in M$, then $x, x \to y \in G$, for some $G \in \mathfrak{F}$, thus $y \in G \subseteq M$. Therefore M is a filter and also proper, since no $G \in \mathfrak{F}$ contains 0, thus $0 \notin M$. By Proposition 3.5, M is prime of type (II) and by Remark 3.1, M is prime of type (I) and obviously maximal.* □

Proposition 3.11 (i) *Let A be prelinear. Then any prime filter of type (I) can be extended to a unique maximal filter,*

(ii) *Any maximal filter of A is a prime filter of type (I).*

Proof:

(i) *It proves by Propositions 3.9 and 3.10.*

(ii) *Let M be a maximal filter of A. Since M is proper and by Corollary 3.3, there exists a prime filter P of type (I) such that $M \subseteq P$ that implies $M = P$.* □

In Proposition 3.11, prelinear condition is necessary because:

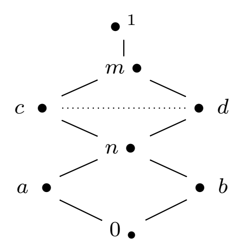
Example 3.9 *Let $A = \{0, a, b, n, c, d, m, 1\}$ be a skew lattice such that $0 < a, b < n < c, d < m < 1$, $c \,\mathbb{D}\, d$ and $\mathbb{D}_c = \{c, d\}$. $A = (A, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated skew lattice with 0, with the following operations:*

$\to$	0	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
n	0	a	b	1	1	1	1	1
c	0	a	b	$\mathbb{D}_c$	1	1	1	1
d	0	a	b	$\mathbb{D}_c$	1	1	1	1
m	0	a	b	n	$\mathbb{D}_c$	$\mathbb{D}_c$	1	1
1	0	a	b	n	$\mathbb{D}_c$	$\mathbb{D}_c$	m	1

$\odot$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	n	n	c	c
d	0	a	b	n	n	n	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1

$\vee$	0	a	b	n	c	d	m	1
0	0	a	b	n	c	d	m	1
a	a	a	n	n	c	d	m	1
b	b	n	b	n	c	d	m	1
n	n	n	n	n	c	d	m	1
c	c	c	c	c	c	d	m	1
d	d	d	d	d	c	d	m	1
m	m	m	m	m	m	m	m	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	c	c	c	c
d	0	a	b	n	d	d	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1



In this example, A is not prelinear ($(a \to b) \vee (b \to a) \vee (a \to b) = n \neq 1$), $P = \{1\}$ is a prime filter of type (I) and $M_1 = \{b, n, c, d, m, 1\}, M_2 = \{a, n, c, d, m, 1\}$ are maximal filters of A and $P \subseteq M_1, M_2$ i.e. if A is not prelinear, then for prime filter P of type (I), maximal filter M contains P is not unique.

4 Radical of a Filter

The intersection of all the maximal filters (maximal deductive systems) of A is called the radical of A and will be denoted by $Rad(A)$ and intersection of all the maximal filters of A that contain F is called the radical of F and will be denoted by $Rad(F)$. It is obvious that $Rad(A), Rad(F) \in Ds(A)$.

Example 4.1 *In Example 3.1, $Rad(A) = A \setminus \{0', 0\}$ and $Rad(\{n, c, d, 1\}) = A \setminus \{0', 0\}$.*

Theorem 4.1 *Let F be a filter of prelinear residuated skew lattice A with 0. Then*

$$Rad(F) = \{x \in A | (x^n)^* \to x \subseteq F, \; for \; all \; n \in N\}.$$

Proof: *If $F = A$, then $Rad(A) = A = \{x \in A | (x^n)^* \to x \subseteq A, for \; all \; n \in N\}$. Now, we assume that $F \neq A$. Let $x \in Rad(F)$ and there is a $k \in N$ such that $(x^k)^* \to x \nsubseteq F$. Then by Proposition 3.8 and Remark 3.1, there exists a prime filter P of type (II) such that $F \subseteq P$ and $(x^k)^* \to x \nsubseteq P$. Since P is a prime filter of type (II), we obtain $x \to (x^k)^* \subseteq P$. Also by Proposition 3.11, there exists a maximal filter M of A such that $P \subseteq M$. Therefore $x \to (x^k)^* \subseteq M$. If $x \in M$, then $x^n \in M$, for all $n \in N$, thus in particular $x^k \in M$. Also we have $x \to (x^k)^* \subseteq M$, hence $(x^k)^* \subseteq M$ and so $0 = x^k \odot (x^k)^* \in M$, which is a contradiction. Thus $x \notin M$. We have $F \subseteq P \subseteq M$ and $x \notin M$, hence $x \notin Rad(F)$, which is a contradiction. Therefore $(x^k)^* \to x \subseteq F$, for all $k \in N$.*

Conversely, let $(x^n)^* \to x \subseteq F$, for all $n \in N$ and $x \notin Rad(F)$. Then there exists a maximal filter M of A, such that $F \subseteq M$ and $x \notin M$. Since M is a maximal filter of A, then there exists $n \in N$ such that $(x^n)^* \subseteq M$. We have $(x^n)^* \to x \subseteq F \subseteq M$, hence $x \in M$. Then $x^n \in M$ and so $0 = (x^n)^* \odot x^n \in M$, which is a contradiction. □

Proposition 4.1 *Let A be a prelinear residuated skew lattice with 0. Then*

$$Rad(A) = \{x \in A | (x^n)^* \preceq x\}.$$

Proof: *Let $B = \{x \in A | (x^n)^* \preceq x\}$. Suppose that $x \notin B$ i.e. there exists $n \in N$ such that $(x^n)^* \npreceq x$. Therefore $(x^n)^* \to x \neq 1$ which implies that there exists a prime filter P of type (II) such that $(x^n)^* \to x \nsubseteq P$ thus $x \to (x^n)^* \in P$. By Zorn Lemma, there exists a maximal filter M such that $P \subseteq M$, therefore $x \to (x^n)^* \subseteq M$. If $x \in M$, then $x^n \in M$ that implies $(x^n)^* \subseteq M$, which is a contradiction. Thus $x \notin M$ so $x \notin \bigcap_{M \in Max(A)} M = Rad(A)$, i.e. $Rad(A) \subseteq B$.*

Conversely, if $x \notin \bigcap_{M \in Max(A)} M = Rad(A)$, then there exists a maximal filter M such that $x \notin M$. Therefore $(x^n)^* \subseteq M$, if $(x^n)^* \preceq x$, then $x \in M$ which is a contradiction. Thus $(x^n)^* \npreceq x$ i.e. $x \notin B$, so $B \subseteq Rad(A)$.
□

Theorem 4.2 *Let A be a prelinear residuated skew lattice with 0 and F be a proper filter of A and $x \in A$. Then $x \in Rad(F)$ iff $x^* \to x^n \subseteq F$, for all $n \in N$.*

Proof: *Let $x \in Rad(F)$. Since $Rad(F)$ is a filter of A, we obtain $x^n \in Rad(F)$, for all $n \in N$. So by Theorem 4.1, $((x^n)^m)^* \to x^n \subseteq F$, for all $m \in N$. We have $x^{nm} \preceq x$ then $(x^{nm})^* \to x^n \preceq x^* \to x^n$. Therefore $x^* \to x^n \subseteq F$, for all $n \in N$.*

Conversely, let $x^* \to x^n \subseteq F$, for all $n \in N$ and $x \notin Rad(F)$. Then there exists a maximal filter M of A such that $F \subseteq M$ and $x \notin M$. Hence there exists $m \in N$ such that $(x^m)^* \subseteq M$. We have $x^m \preceq (x^m)^{**}$, hence $x^* \to x^m \preceq x^* \to (x^m)^{**}$, and so $x^* \to (x^m)^{**} \subseteq F$. We have

$$(x^m)^* \to x^{**} = (x^m)^* \to (x^* \to 0) \,\mathbb{D}\, x^* \to ((x^m)^* \to 0) \quad = x^* \to (x^m)^{**} \subseteq F.$$

Therefore $(x^m)^* \to x^{**} \subseteq F \subseteq M$. Since $(x^m)^* \subseteq M$, we get that $x^{**} \subseteq M$ and so $(x^{**})^n \subseteq M$, for all $n \in N$, thus in particular $(x^{**})^m \subseteq M$. We have $(x^m)^{**} \succeq (x^{**})^m \subseteq M$, thus $(x^m)^{**} \subseteq M$. Since $(x^m)^* \subseteq M$, hence $0 = (x^m)^{**} \odot (x^m)^* \in M$, which is a contradiction. □

For filters F_1, F_2 of A we define $F_1 \to F_2 = \{x \in A | F_1 \cap [x) \subseteq F_2\}$.

Theorem 4.3 *If F, F_i, G are proper filters of A, then following conditions are satisfied:*

- (i) $F \subseteq Rad(F)$,
- (ii) If $F \subseteq G$, then $Rad(F) \subseteq Rad(G)$,
- (iii) If M is a maximal filter of A, then $Rad(M) = M$,
- (iv) $Rad(Rad(F)) = Rad(F)$,
- (v) If $< F \cup G >$ is a proper filter of A, then $Rad(F) \cup Rad(G) \subseteq Rad < F \cup G >$, $(< F \cup G >= \{x \in A | x \succeq d_1 \odot d_2, d_1 \in F, d_2 \in G\})$,
- (vi) $Rad(A) \subseteq Rad(F)$,
- (vii) $Rad(\bigcup_{i\in I} F_i) = \bigcup_{i\in I} Rad(F_i)$,
- (viii) $Rad(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} Rad(F_i)$,
- (ix) $Rad(\{1/F\}) = Rad(F)/F$,
- (x) $Rad(F) \to Rad(G) \subseteq F \to Rad(G)$,
- (xi) $Rad(F \to G) \subseteq Rad(F \to Rad(G))$,
- (xii) If $x, y \in Rad(F)$, then $x^* \to y \subseteq F$,
- (xiii) If $x, y \in Rad(F)$, then $(x^* \odot y^*)^* \subseteq F$,
- (xiv) $x^* = 0$, for all $x \in A \setminus \{0\}$, iff $Rad(F) = A \setminus \{0\}$,
- (xv) If $x \in Rad(F)$, then $((x^n)^* \odot (x^n)^*) \to x = 1$, for all $n \in N$,
- (xvi) $Rad(F) = A$ iff $F = A$.

In $vi - viii$ and $xi - xvi$, A is a prelinear residuated skew lattice with 0.

Proof:

(i) *is clear.*

(ii) *If* $x \in Rad(F)$, *then* $x \in \bigcap_{F \subseteq M \in Max(A)} M$.
Therefore $x \in \bigcap_{G \subseteq M \in Max(A)} M$ *i.e.* $x \in Rad(G)$.

(iii) *is clear.*

(iv) *According to* (ii), *we have* $Rad(F) \subseteq Rad(Rad(F))$ *(since* $F \subseteq Rad(F)$*). Suppose* $x \in Rad(Rad(F))$, *therefore* $x \in M$, *for all maximal filters* M *of* A *that* $Rad(F) \subseteq M$. *Let* M_0 *be an arbitrary maximal filter of* A *that contains* F. *Then* $Rad(F) \subseteq Rad(M_0) = M_0$, *therefore* $x \in M_0$. *So* $x \in Rad(F)$. *Hence* $Rad(Rad(F)) = Rad(F)$.

(v) *Since* $F, G \subseteq < F \cup G >$, *then* $Rad(F), Rad(G) \subseteq Rad < F \cup G >$ *(according to (ii)). Therefore* $Rad(F) \cup Rad(G) \subseteq Rad < F \cup G >$.

(vi) *By Theorem 4.1 and Proposition 4.1, it is clear.*

(vii) *If* $x \in Rad(\bigcup_{i \in I} F_i)$, *then* $(x^n)^* \to x \subseteq \bigcup_{i \in I} F_i$ *so there exists* $i \in I$ *such that* $(x^n)^* \to x \subseteq F_i$ *therefore* $x \in Rad(F_i)$. *On the other hand since* $F_i \subseteq \bigcup_{i \in I} F_i$, *then* $Rad(F_i) \subseteq Rad(\bigcup_{i \in I} F_i)$ *for all* $i \in I$, *hence* $\bigcup_{i \in I} Rad(F_i) \subseteq Rad(\bigcup_{i \in I} F_i)$.

(viii) *Since* $\bigcap_{i \in I} F_i \subseteq F_i$ *for all* $(i \in I)$, *and* (ii), *it is clear. Conversely, let* $x \in \bigcap_{i \in I}(Rad(F_i))$. *Then* $x \in Rad(F_i)$, *for all* $i \in I$, *and so* $(x^n)^* \to x \subseteq F_i$, *for all* $i \in I$ *and* $n \in N$. *Hence* $(x^n)^* \to x \subseteq \bigcap_{i \in I} F_i$, *for all* $n \in N$, *that is* $x \in Rad(\bigcap_{i \in I} F_i)$. *Therefore* $Rad(\bigcap_{i \in I} F_i) = \bigcap_{i \in I} Rad(F_i)$.

(ix) *By definition of radical we have* $Rad(\{1/F\}) = \bigcap_{F \subseteq M \in Max(A)}(M/F)$ $= (\bigcap_{F \subseteq M \in Max(A)} M)/F = Rad(F)/F$.

(x) *Let* $x \in Rad(F) \to Rad(G)$. *Then* $Rad(F) \cap [x) \subseteq Rad(G)$. *Hence* $F \cap [x) \subseteq Rad(G)$, *that is* $x \in F \to Rad(G)$.

(xi) *Let* $x \in Rad(F \to G)$. *Then* $(x^n)^* \to x \subseteq F \to G$, *for all* $n \in N$, *that is* $F \cap [(x^n)^* \to x) \subseteq G \subseteq Rad(G)$, *for all* $n \in N$. *Hence* $(x^n)^* \to x \subseteq F \to Rad(G)$, *for all* $n \in N$, *and so* $x \in Rad(F \to Rad(G))$.

(xii) *Let* $x, y \in Rad(F)$. *Then* $x \odot y \in Rad(F)$ *and so* $(x \odot y)^* \to (x \odot y) \subseteq F$. *We have* $x \odot y \preceq x$ *then* $(x \odot y)^* \to (x \odot y) \preceq x^* \to (x \odot y)$. *Therefore* $x^* \to (x \odot y) \subseteq F$. *Since* $x \odot y \preceq y$, *then* $x \odot y \to y = 1 \in F$ *and so* $(x^* \to (x \odot y)) \odot ((x \odot y) \to y) \in F$. *Thus by Lemma 2.3* (10), *we obtain* $x^* \to y \subseteq F$.

(xiii) *Let* $x, y \in Rad(F)$. *Then by* (xii), *we have* $x^* \to y \subseteq F$. *Since* $y \preceq y^{**}$ *we get that* $x^* \to y \preceq x^* \to y^{**}$ *and so* $x^* \to y^{**} \subseteq F$. *Thus by Theorem* 3.2 *of [2], we have* $(x^* \odot y^*)^* \subseteq F$.

(xiv) *Let* $x^* = 0$, *for all* $x \in A \setminus \{0\}$. *It is clear that* $Rad(F) \subseteq A \setminus \{0\}$. *We must show that* $A \setminus \{0\} \subseteq Rad(F)$. *Take* $x \in A \setminus \{0\}$, *then by hypothesis* $x^* = 0$ *and so* $x^* \to x^n = 0 \to x^n = 1 \in F$, *for all* $n \in N$. *Therefore*

$x \in Rad(F)$ by Theorem 4.2, Hence $Rad(F) = A \setminus \{0\}$. Conversely, let $Rad(F) = A \setminus \{0\}$ and there exists $x \in A \setminus \{0\}$ such that $x^* \neq 0$. Hence by hypothesis $x^* \subseteq Rad(F), x \in Rad(F)$, so $0 \in Rad(F)$, which is a contradiction.

(*xv*) *Let $x \in Rad(F)$. Then $(x^n)^* \to x \preceq (x^n)^*$ or $(x^n)^* \preceq (x^n)^* \to x$, for all $n \in N$. Let $(x^n)^* \to x \preceq (x^n)^*$. Since $(x^n)^* \to x \subseteq F$, then $(x^n)^* \subseteq F$ and so $x \in F$. Hence $x^n \in F$, for all $n \in N$, so $(x^n)^* \odot x^n \in F$. Therefore $0 \in F$, which is a contradiction. Hence $(x^n)^* \preceq (x^n)^* \to x$, for all $n \in N$. Then $(x^n)^* \to ((x^n)^* \to x) = 1$, for all $n \in N$, so $((x^n)^* \odot (x^n)^*) \to x = 1$, for all $n \in N$.*

(*xvi*) *Let $Rad(F) = A$. Then $0 \in Rad(F)$ and so $0 = 1 \to 0 = (0^n)^* \to 0 \in F$, for all $n \in N$. Therefore $F = A$. The converse is clear.* □

According to Theorem 4.3, $Rad(F)$ is a closure operator.

Definition 4.1 *An element x of a residuated skew lattice A with 0 is called infinitesimal if $x \neq 1$ and $x^n \succeq x^*$, for any $n \geq 1$. We denote by $Inf(A)$ the set of all infinitesimal elements of A.*

Example 4.2 *In Example 3.6, $Inf(A) = \{n, c, d, m\}$.*

Proposition 4.2 *For every nonunit element x of a prelinear residuated skew lattice A with 0, $(x \neq 1)$, $x \in Rad(A)$ iff $x \in Inf(A)$.*

Proof: *Let $x \in Rad(A)$. Then by Proposition 4.1, $(x^n)^* \preceq x$. Therefore for $n = 1$, $x^* \preceq x$. Since $x^n \in Rad(A)$, then $(x^n)^* \preceq x^n$. On the other hand $x^* \odot x^n \preceq x^* \odot x = 0$ which implies $x^* \odot x^n = 0$, for all $n \in N$, implies $x^* \preceq (x^n)^*$. Thus $x^* \preceq x^n$.*

Conversely, let x be an infinitesimal and suppose $x \notin Rad(A)$. Thus there is a maximal filter M of A such that $x \notin M$. By Corollary 2.2, there is $n \geq 1$ such that $(x^n)^* \subseteq M$. By hypothesis $x^* \preceq x^n$ hence $(x^n)^* \preceq x^{**}$, so $x^{**} \subseteq M$. On the other hand $(x^{**})^n \preceq (x^n)^{**}$, hence $(x^n)^{**} \subseteq M$ therefore $0 = (x^n)^* \odot (x^n)^{**} \in M$, that is a contradiction. □

Corollary 4.1 *$Rad(A) \setminus \{1\} = Inf(A)$, in prelinear residuated skew lattice A with 0.*

Example 4.3 *In Example 3.4, $Inf(A) = \{a, b\} = Rad(A) \setminus \{1\} = \{a, b\}$.*

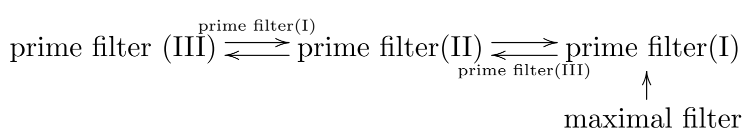
Remark 4.1 *In branchwise residuated skew lattice A, $Inf(\uparrow u) \subseteq Rad(\uparrow u)$ for all $u \in A$.*

Proof: *Since any branch of branchwise residuated skew lattice A is a (distributive) residuated lattice, then it is clear by Corollary 1.68 of [14].* □

In fact, in branchwise residuated skew lattice we have $Inf(\uparrow u) \subseteq Rad(\uparrow u)$, for all $u \in A$, but in non conormal prelinear residuated skew lattice A with 0, $Rad(A) \setminus \{1\} = Inf(A)$.

5 Conclusion and Future Research

In this paper, the relation between skew deductive system and skew filter were studied. In residuated skew lattice and branchwise residuated skew lattice, deductive system and filter were equivalent. Results showed that in conormal residuated skew lattice which $x \to y = y \vee x \vee y \to_y y$, skew deductive system is a skew filter and deductive system and skew deductive system are equivalent. Some types of prime (skew) filters were defined in residuated skew lattices and relations among them were investigated and the concept of prime filters in residuated lattices was generalized by them. We showed the relationship between prime (skew) filters and prelinear residuated skew lattices and by a example was shown that extention of a prime filter of type (I) is not a prime filter of type (I). The notion of the radical of a filter F was introduced and was presented a characterization and many important properties of $Rad(F)$. In non conormal prelinear residuated skew lattice with 0 was shown that $Rad(A) \setminus \{1\} = Inf(A)$. In the future, we study the prime spectrum of a residuated skew lattice and study this topological space and we obtain some properties of it. We try to define some types of (skew) filters, investigate the relationship among them. We consider some relationships between those filters and quotient algebras that are constructed via those filters and classification of residuated skew lattices. A primary decomposition for the filters will be gotten and some properties of the filters based on the prime filters will be gotten too. Filters can be used to derive topological properties of algebraic structures and it is possible to define different topologies on the algebraic structures by using filters, which we address in future. We can get some algebraic properties by studying the topologies too. Various topologies will be defined as Zariski topology and Flat topology on the residuated skew lattice and their properties will be obtained.



The relationship among some types of filters in residuated skew lattices

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