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An Optimum Lower Bound for the Weights of Maximum Weight Matching in Bipartite Graphs

Shibsankar DAS[1]

Abstract

The problem of computing a maximum weight matching in a bipartite graph is one of the fundamental algorithmic problems that has played an important role in the development of combinatorial optimization and algorithmics. Let $\mathcal{G}_{w,\sigma}$ is a collection of all weighted bipartite graphs, each having σ and w as the size of each of the non-empty subset of the vertex partition and the total weight of the graph, respectively. We give a tight lower bound $\lceil \frac{w-\sigma}{\sigma} \rceil + 1$ for the set $\{ Wt(mwm(G)) \mid G \in \mathcal{G}_{w,\sigma} \}$ which denotes the collection of weights of maximum weight bipartite matchings of all the graphs in $\mathcal{G}_{w,\sigma}$.

Keywords: Bipartite graph, Maximum weight bipartite matching, Lower bound for weights of bipartite matching, Combinatorial optimization, String matching.

1 Introduction

We use the notations $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive and non-negative integers, respectively. All the graphs considered in this paper are simple, undirected and connected. Let $G = (V = V_1 \cup V_2, E, Wt)$ be a *weighted bipartite graph* where V_1 and V_2 are two disjoint independent non-empty subsets of the vertex set V of G, and each edge in the edge set E of G

[1]Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi - 221005, Uttar Pradesh, India, E-mail: shib.iitm@gmail.com, shibsankar@bhu.ac.in

connects a vertex in V_1 to a vertex in V_2, with positive integer weight given by the weight function $Wt\colon E \rightarrow \mathbb{N}$. Let the *total weight* of G be denoted by w and is defined by $w = Wt(G) = \sum_{e \in E} Wt(e)$. For uniformity, let us treat an unweighted graph as a weighted graph having unit weight for all of its edges.

We use an unordered pair notation $\{u, v\}$ for an edge $e \in E$ joining the two vertices $u \in V_1$ and $v \in V_2$, and its weight is denoted by $Wt(e) = Wt(u, v)$. Further, an edge $e = \{u, v\}$ is said to be *incident on* the vertices u and v, and the vertices u and v are each said to be *incident to* e. Two vertices $u, v \in V$ of G are *adjacent* in G if there exists an edge $e = \{u, v\} \in E$ of G to which both vertices are incident. Two edges $e_1, e_2 \in E$ of G are *adjacent* if there exists a vertex $v \in V$ to which they are both incidents on [4].

1.1 Basics of Maximum Weight Bipartite Matching

A subset $M \subseteq E$ of edges of a graph G is a *matching* of G if no two edges of M share a common vertex. A vertex $v \in V$ is said to be *covered* or *matched* by the matching M if it is incident to an edge of M; otherwise, v is *unmatched* [3]. A matching M of G is called a *Maximum (Cardinality) Matching* (MCM) if there does not exist any other matching of G with greater cardinality. We denote such matching by $mm(G)$. The weight of a matching M is defined as $Wt(M) = \sum_{e \in M} Wt(e)$. A matching M of G is a *Maximum Weight Matching* (MWM), denoted as $mwm(G)$, if $Wt(M) \geq Wt(M')$ for every other matching M' of the graph G. Observe that if G is an unweighted graph then a $mwm(G)$ is a $mm(G)$, which we write as $mwm(G) = mm(G)$ in short and its weight is given by $Wt(mwm(G)) = |mm(G)|$. Similarly, if G is an undirected and weighted graph with $Wt(e) = c$ for all edges e in G and c is a constant then also we have $mwm(G) = mm(G)$ with the weight of the matching as $Wt(mwm(G)) = c * |mm(G)|$. Given a bipartite graph G, the *Maximum Weight Bipartite Matching* (MWBM) problem computes a MWM of G.

The problem of MWBM is a well-studied key problem in combinatorial optimization and algorithmics, and has a wide range of applications (see textbooks [13, 16]). Several algorithms have been proposed for computing MWBM, improving both theoretical and practical running times. A complexity survey of some of the well known exact algorithms for the MWBM problem is summarized in [6]. Moreover, several randomized and approximate algorithms are also proposed to solve this problem, see for example [8, 15].

A peek through similar research papers in this field suggests that most existing works focus on the lower bounds for the size of the matching rather than the weight of the matching in graphs. For example, lower bounds on the size of maximum matchings are studied in graphs with small maximum degree and 3-connected planar graphs [2,14], connected k-regular ($k \geq 3$) graphs of order n [11], hypergraphs of rank three [12], subcubic graphs [9], etc.

1.2 Our Contribution

In this paper, we give a tight lower bound for the weights of MWBM in bipartite graphs having fixed weight and vertex size. Let $\mathcal{G}_{w,\sigma}$ is the collection of all weighted bipartite graphs, each of whose weight is w and σ is the size of each of the two non-empty subsets of the vertex partition. The set of weights of MWBM of the graphs in $\mathcal{G}_{w,\sigma}$ is denoted by $\{ Wt(mwm(G)) \mid G \in \mathcal{G}_{w,\sigma}\}$. We prove that $\lceil \frac{w-\sigma}{\sigma} \rceil + 1$ is a lower bound of $\{ Wt(mwm(G)) \mid G \in \mathcal{G}_{w,\sigma}\}$ and this bound is optimum.

This result can be applied to develop optical packet switches to transform data center scalability [1]. It is also applicable to the area of stringology [7,10]. An equivalent result (see Lemma 3.13 in [5]) of this outcome is applied for enumerating the error classes (see Theorem 3.14 in [5]) in the approximate parameterized string matching. Besides, this finding may be applied in any kind of communication network.

1.3 Roadmap

The rest of the paper is organized as follows. In Section 2, we partition the class of graphs in $\mathcal{G}_{w,\sigma}$ into two subclasses and provide a tight lower bound for the weights of MWBM of graphs in each of the subclasses of $\mathcal{G}_{w,\sigma}$. A summary is given in Section 3.

2 A Tight Lower Bound for the Weights of Maximum Weight Bipartite Matching in $\mathcal{G}_{w,\sigma}$

Let $\mathcal{G}_{w,\sigma}$ denotes the collection of all weighted bipartite graphs, each of whose total weight is fixed to w and σ is the size of each of the two non-empty subsets of the vertex partition of any graph in $\mathcal{G}_{w,\sigma}$. Let us assume that Σ_P and Σ_T are the pair of non-empty subsets of the vertex set of each of the bipartite graphs in $\mathcal{G}_{w,\sigma}$. Therefore, $|\Sigma_P| = |\Sigma_T| = \sigma$.

We now define two partition classes $\mathcal{G}_{w\geq\sigma}$ and $\mathcal{G}_{w<\sigma}$ based on the relation between the fixed values w and σ. They are as follows.

$$\mathcal{G}_{w\geq\sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|,\ w = Wt(G),\ \text{and}\ w \geq \sigma\}$$

and

$$\mathcal{G}_{w<\sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|,\ w = Wt(G),\ \text{and}\ w < \sigma\}.$$

Observe that based on the above construction we can redefine $\mathcal{G}_{w,\sigma}$ as

$$\mathcal{G}_{w,\sigma} = \begin{cases} \mathcal{G}_{w\geq\sigma}, & \text{if } w \geq \sigma. \\ \mathcal{G}_{w<\sigma}, & \text{if } w < \sigma. \end{cases}$$

Now we prove that the value of $\min_{G\in\mathcal{G}_{w,\sigma}}\{Wt(mwm(G))\}$, which denotes the minimum weight among the maximum weight bipartite matchings of all the graphs in $\mathcal{G}_{w,\sigma}$, is $\lceil \frac{w-\sigma}{\sigma} \rceil + 1$.

Let us first prove it for $\mathcal{G}_{w\geq\sigma}$. Since $w \geq \sigma$, we can always write the term w as $q\sigma + r$ for some $q, r \in \mathbb{N}_0$, where $0 < r \leq \sigma$. First, we show the existence of bipartite graph $G \in \mathcal{G}_{w\geq\sigma}$ such that $Wt(mwm(G)) = q + 1$. We then prove in Theorem 2 that $q + 1$ is the tight lower bound of the set $\{Wt(mwm(G)) \mid G \in \mathcal{G}_{w\geq\sigma}\}$.

Theorem 1 *Let* $\mathcal{G}_{w\geq\sigma} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, w = Wt(G)$ *and* $w \geq \sigma\}$. *If* $w = q\sigma + r$ *for some non-negative integers* q *and* r *where* $0 < r \leq \sigma$, *then there exists a bipartite graph* $G \in \mathcal{G}_{w\geq\sigma}$ *such that* $Wt(mwm(G)) = q + 1$.

Proof: For the case $q = 0$, we have $w = q\sigma + r = r = \sigma$ as $0 < r \leq \sigma$ and $w \geq \sigma$. Figure 1(a) shows a bipartite graph $G' = (\Sigma_P \cup \Sigma_T, E', Wt) \in \mathcal{G}_{w\geq\sigma}$ for this case. The weight of the graph is $Wt(G') = \sigma$. In this graph G', $Wt(mwm(G')) = 1 = q + 1$.

For $q \geq 1$, the total weight of any bipartite graph in $\mathcal{G}_{w\geq\sigma}$ is $w = q\sigma + r$. We produce such a bipartite graph $G'' \in \mathcal{G}_{w\geq\sigma}$ as shown in Figure 1(b) with $Wt(mwm(G'')) = q + 1$. □

Observe that in a weighted graph G, any edge e of weight $c \in \mathbb{N}$ can be thought of as c number of overlapping unit weight edges. Similarly, increasing the weight of a bipartite graph G by adding a weight $c \in \mathbb{N}$ is equivalent to adding c unit weight edges in G. Without loss of generality, we assume these as a convention while incrementing weight in a weighted graph.

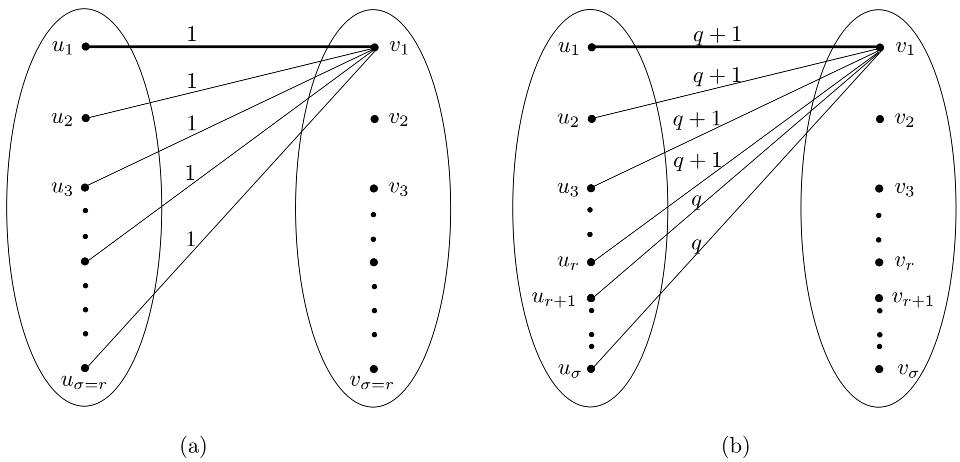

Figure 1: Existence of bipartite graph $G \in \mathcal{G}_{w\geq\sigma}$, such that $Wt(mwm(G)) = q+1$ where $w = q\sigma + r$ for some $q, r \in \mathbb{N}_0$ and $0 < r \leq \sigma$. **(a)** An example of bipartite graph $G' \in \mathcal{G}_{w\geq\sigma}$ for the case $q = 0$. **(b)** An example of bipartite graph $G'' \in \mathcal{G}_{w\geq\sigma}$ for the case $q \geq 1$. In both graphs, the thick edge represents a maximum weight matching edge.

Theorem 2 (Tight Lower Bound for the Weights of MWBMs) *Let* $\mathcal{G}_{w\geq\sigma} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|,\ w = Wt(G)$ *and* $w \geq \sigma\}$. *Then*

$$\min_{G\in\mathcal{G}_{w\geq\sigma}} \{ Wt(mwm(G))\} = q+1$$

where $w = q\sigma + r$ *for some non-negative integers* q *and* r*, and* $0 < r \leq \sigma$.

Proof: For $\sigma = 1$, the statement is trivially true. So we consider $\sigma \geq 2$ and prove the statement $\min_{G\in\mathcal{G}_{w\geq\sigma}}\{Wt(mwm(G))\} = q+1$ by using the principle of mathematical induction on $q \in \mathbb{N}_0$. Let $\Sigma_P = \{u_1, u_2, \ldots, u_\sigma\}$ and $\Sigma_T = \{v_1, v_2, \ldots, v_\sigma\}$ are the disjoint vertex sets of the graphs in $\mathcal{G}_{w\geq\sigma}$. For simplicity, we denote $\mathcal{G}_{q+1} = \mathcal{G}_{w\geq\sigma}$ when $w = q\sigma + r$ for some $q, r \in \mathbb{N}_0$ where $0 < r \leq \sigma$, that is, $q = \lceil \frac{w-\sigma}{\sigma} \rceil$ where q is represented as a function of w and σ only.

***Base Step*:** Let $q = 0$. Then $w = r = \sigma$ because $0 < r \leq \sigma$ and $w \geq \sigma$, and

$$\mathcal{G}_1 = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, Wt(G) = \sigma\}.$$

Since for any graph $G = (\Sigma_P \cup \Sigma_T, E, Wt) \in \mathcal{G}_1$, $|\Sigma_P| = |\Sigma_T| = \sigma$ and $Wt(G) = \sigma$, therefore $\min_{G\in\mathcal{G}_1}\{Wt(mwm(G))\} = 1 = q+1$.

***Induction Hypothesis*:** Assume that for $q = i$, $\min_{G\in\mathcal{G}_{i+1}}\{Wt(mwm(G))\} = i+1$, where

$$w = i\sigma + r, \text{ and}$$
$$\mathcal{G}_{i+1} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, Wt(G) = i\sigma + r\}.$$

Let $\mathcal{G}'_{i+1} = \{G \in \mathcal{G}_{i+1} \mid Wt(mwm(G)) = i+1\}$. The set $\mathcal{G}'_{i+1}$ is non-empty by Theorem 1. We use this set in the following inductive step.

***Inductive Step*:** Let $q = i+1$. We have to prove $\min_{G\in\mathcal{G}_{i+2}}\{Wt(mwm(G))\} = i+2$, where

$$w = (i+1)\sigma + r, \text{ and}$$
$$\mathcal{G}_{i+2} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, Wt(G) = (i+1)\sigma + r\}.$$

The existence of a graph $G \in \mathcal{G}_{i+2}$ with $Wt(mwm(G)) = i+2$ is shown in Theorem 1. Therefore, we only have to prove that there does not exist any graph in $\mathcal{G}_{i+2}$ whose weight of a maximum weight matching is $i+1$. Let us prove it by contradiction. Suppose there exists a graph $G_* \in \mathcal{G}_{i+2}$ such that $Wt(mwm(G_*)) = i+1$.

Observe that, for any graph in $\mathcal{G}_{i+2}$, its weight is equal to $w = (i+1)\sigma + r = (i\sigma + r) + \sigma$. Therefore, any graph in $\mathcal{G}_{i+2}$ is generated by adding a total of σ weight to the non-negative weight edges of a graph in $\mathcal{G}_{i+1}$.

Therefore, G_* can only be constructed from a graph in $\mathcal{G}'_{i+1}$ by adding a total of σ weight to the non-negative weight edges of that graph in $\mathcal{G}'_{i+1}$; because for all $G \in \mathcal{G}_{i+1} \setminus \mathcal{G}'_{i+1}$, $Wt(mwm(G)) > i+1$. Let $\Sigma = \{e_1, e_2, e_3, \ldots\}$ is the edges, where $\sigma = \sum_{e_i\in\Sigma} Wt(e_i)$, whose weights are increased in $G \in \mathcal{G}'_{i+1}$ to build G_*.

Case 1: Let $G \in \mathcal{G}'_{i+1}$ and $M = mwm(G)$. If there exists at least one edge e in Σ such that $e \in M$ or if both end points of e are unmatched vertices, then let $M' = M \cup \{e\}$, which is a weighted matching of G_*, not necessarily of maximum weight. Therefore

$$Wt(mwm(G_*)) \geq Wt(M') = Wt(M) + Wt(e) = i+1 + Wt(e) > i+1$$

which is a contradiction because we assumed that $Wt(mwm(G_*)) = i + 1$.

Note: Hence for the rest of the cases, we assume that none of the edges in Σ, which are added in $G \in \mathcal{G}'_{i+1}$ to get the $G_* \in \mathcal{G}_{i+2}$, belongs to M; or both end points of none of the edges in Σ are unmatched vertices. Therefore if $e = \{u, v\} \in \Sigma$, then: (a) u is an unmatched vertex and v is a matched vertex or vice versa, or (b) both u and v are matched vertices, but $e \notin M = mwm(G)$.

Case 2: Let there exists at least one edge $e = \{u, v\} \in \Sigma$ such that $Wt(e) = w_\sigma \geq 2$. Then we have the following two sub-cases which are shown in Figure 2. Let $G \in \mathcal{G}'_{i+1}$ and $M = mwm(G)$.

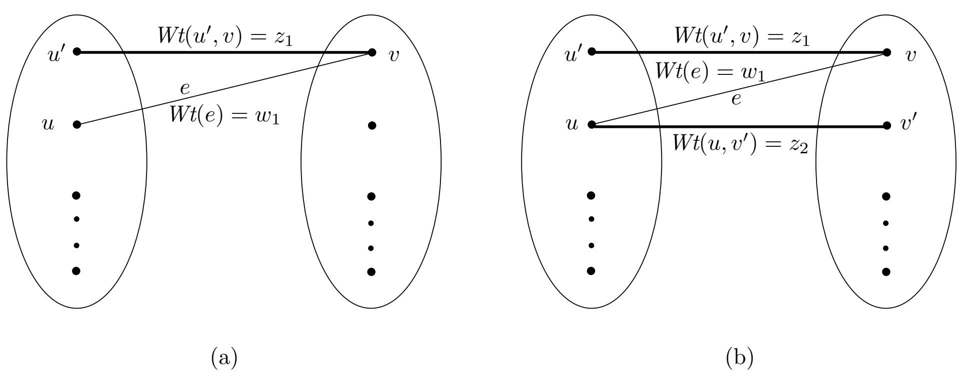

Figure 2: **(a)** This graph gives a pictorial representation of the Sub-case 2(a) in Theorem 2. **(b)** Sketch of the graph considered in Sub-case 2(b) is shown here. In both graphs, the thick edges are maximum weight matching edges.

Sub-case 2(a): Assume that u and v be the unmatched and matched vertices in $G \in \mathcal{G}'_{i+1}$, respectively. So there exists an edge $e' = \{u', v\} \in M$ which is incident on the matched vertex v. Let $Wt(e') = Wt(u', v) = z_1$ and $Wt(e) = Wt(u, v) = w_1$ in the G. Therefore $z_1 \geq w_1$. Now add the edge e (or increase the edge weight of e) in G where $Wt(e) = w_\sigma \geq 2$ in order to generate $G_* \in \mathcal{G}_{i+2}$ such that $Wt(mwm(G_*)) = i + 1$.

If $z_1 < w_1 + w_\sigma$, then let

$$M' = M \setminus \{e'\} \cup \{e\}$$

which is a weighted matching of G_*. Hence

$$\begin{aligned} Wt(mwm(G_*)) &\geq Wt(M') \\ &= Wt(M) - z_1 + w_1 + w_\sigma \\ &= (i+1) - z_1 + w_1 + w_\sigma \\ &> i+1 \end{aligned}$$

which is a contradiction.

Or else,

$$z_1 \geq w_1 + w_\sigma \;\Leftrightarrow\; z_1 - 1 \geq w_1 + (w_\sigma - 1).$$

Therefore we can construct a new graph G' from G by decreasing one unit weight of the edge $e' = \{u', v\} \in M$ and increasing the weight of the edge $e = \{u, v\} \notin M$ by one unit in G. As a consequence, the weight of G' remains the same as that of $G \in \mathcal{G}'_{i+1}$ and so $G' \in \mathcal{G}_{i+1}$. But

$$Wt(mwm(G')) = i < Wt(M) = i+1$$

contradicting the induction hypothesis that $\min_{G \in \mathcal{G}_{i+1}}\{Wt(mwm(G))\}$ $= i+1$.

Sub-case 2(b): Suppose both u and v are matched vertices but $e = \{u, v\} \notin M$. See Figure 2(b). So there exist two edges $e' = \{u', v\} \in M$ and $e'' = \{u, v'\} \in M$ which are incident on the matched vertices v and u, respectively. Let $Wt(e') = z_1$, $Wt(e'') = z_2$ and $Wt(e) = w_1$ in $G \in \mathcal{G}'_{i+1}$.

$$\text{So, } z_1 + z_2 \geq w_1 \quad \text{in } G.$$

Now after adding the edge e in G with $Wt(e) = w_\sigma \geq 2$, if

$$z_1 + z_2 < w_1 + w_\sigma,$$

then let

$$M' = M \setminus \{e', e''\} \cup \{e\}$$

which is a weighted matching of G_*. Hence

$$\begin{aligned} Wt(mwm(G_*)) &\geq Wt(M') \\ &= Wt(M) - z_1 - z_2 + w_1 + w_\sigma \\ &= (i+1) - z_1 - z_2 + w_1 + w_\sigma \\ &> i+1 \end{aligned}$$

which is a contradiction.

Or else,

$$z_1 + z_2 \geq w_1 + w_\sigma \;\Leftrightarrow\; (z_1 - 1) + z_2 \geq w_1 + (w_\sigma - 1).$$

Therefore we can construct a new graph G' from G by reducing one unit weight of the edge $e' = \{u', v\} \in M$ and adding one unit weight to the edge $e = \{u, v\} \notin M$ of G. As a consequence, the weight of G' is the same as that of $G \in \mathcal{G}'_{i+1}$ and so $G' \in \mathcal{G}_{i+1}$. But

$$Wt(mwm(G')) = i < Wt(M) = i + 1$$

which contradicts the hypothesis that $\min_{G \in \mathcal{G}_{i+1}}\{Wt(mwm(G))\} = i + 1$.

Case 3: Let for each edge $e \in \Sigma$, $Wt(e) = 1$. Consider $\Sigma = \{e_1 = \{u_1, v_1\}, e_2 = \{u_2, v_2\}, \ldots, e_\sigma = \{u_\sigma, v_\sigma\}\}$ and their respective weights in $G \in \mathcal{G}'_{i+1}$ are given by $\{w_1, w_2, \ldots, w_\sigma\}$. We add these σ number of edges of Σ in $G \in \mathcal{G}'_{i+1}$ to produce a graph $G_* \in \mathcal{G}_{i+2}$ such that $Wt(mwm(G_*)) = i + 1$. Further let $M = mwm(G)$.

Therefore, there must exist two edges in Σ which are not adjacent. Because if not, then all the edges of Σ are adjacent to one vertex. Without loss of generality, suppose $u_1 = u_2 = \cdots = u_\sigma$. See Figure 3 and consider the following two possibilities.

(a) If $u_1 \in \Sigma_P$ is an unmatched vertex in $G \in \mathcal{G}'_{i+1}$, then there must be another unmatched vertex in Σ_T of the graph G, because $\sigma = |\Sigma_P| = |\Sigma_T|$. Say the unmatched vertex is $v_1 \in \Sigma_T$. If we add an edge $\{u_1, v_1\} \in \Sigma$ in $G \in \mathcal{G}'_{i+1}$, then this kind of graph is already addressed in Case 1. Therefore, at most $\sigma - 1$ number of edges of unit weight can be added in G while generating the G_*. This is a contradiction.

(b) Similarly, if $u_1 \in \Sigma_P$ is a matched vertex in $G \in \mathcal{G}'_{i+1}$, then there must be another matched vertex in Σ_T of the graph G. The rest of the argument is similar to the previous unmatched case.

So we assume the two non-adjacent edges be $e_1, e_2 \in \Sigma$. Then a maximum of four edges in M is adjacent to edges $e_1, e_2 \in \Sigma$. Let e'_1, e'_2, e'_3, e'_4

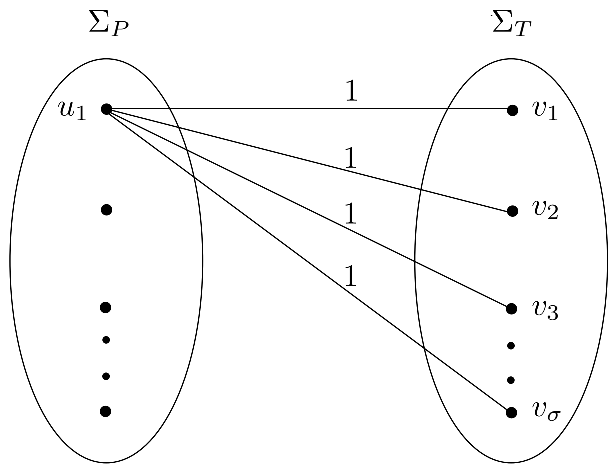

Figure 3: There must exist two edges $e_1, e_2 \in \Sigma$ such that e_1 and e_2 are not adjacent. This kind of graph does not arise in Case 3 of Theorem 2.

are such edges and z_1, z_2, z_3, z_4 are their corresponding weights in $G \in \mathcal{G}'_{i+1}$, respectively. Therefore,

$$z_1 + z_2 + z_3 + z_4 \geq w_1 + w_2 \quad \text{in } G.$$

Now after adding σ edges of Σ in G, if

$$z_1 + z_2 + z_3 + z_4 < w_1 + w_2 + 2,$$

then let

$$M' = M \setminus \{e'_1, e'_2, e'_3, e'_4\} \cup \{e_1, e_2\}$$

which is a weighted matching of G_*. Hence

$$\begin{aligned} Wt(mwm(G_*)) &\geq Wt(M') \\ &= Wt(M) - (z_1 + z_2 + z_3 + z_4) + (w_1 + w_2 + 2) \\ &> i + 1 \end{aligned}$$

which is a contradiction.

Or else,

$$\begin{aligned} & z_1 + z_2 + z_3 + z_4 \geq w_1 + w_2 + 2 \\ \Leftrightarrow\ & z_1 + z_2 + z_3 + z_4 - 1 \geq w_1 + w_2 + 1. \end{aligned}$$

As a consequence, by similar argument as stated in Sub-case 2(b), we can construct a new graph G' whose weight is the same as that of $G \in \mathcal{G}'_{i+1}$ and therefore $G' \in \mathcal{G}_{i+1}$. But

$$Wt(mwm(G')) = i < Wt(M) = i + 1$$

contradicting the induction hypothesis that $\min_{G \in \mathcal{G}_{i+1}}\{Wt(mwm(G))\} = i + 1$.

This completes the proof. □

An equivalent statement of Theorem 2 is the following.

Corollary 1 *For the partition class* $\mathcal{G}_{w \geq \sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, w = Wt(G)$ *and* $w \geq \sigma\}$

$$\min_{G \in \mathcal{G}_{w \geq \sigma}}\{Wt(mwm(G))\} = \left\lceil \frac{w - \sigma}{\sigma} \right\rceil + 1.$$

Proof: Since $w \geq \sigma$, we can always write w as $q\sigma + r$ for some q, $r \in \mathbb{N}_0$ where $0 < r \leq \sigma$. Then the term $\lceil \frac{w-\sigma}{\sigma} \rceil$ can be written as

$$\left\lceil \frac{w - \sigma}{\sigma} \right\rceil = \left\lceil \frac{q\sigma + r - \sigma}{\sigma} \right\rceil = \left\lceil \frac{(q-1)\sigma + r}{\sigma} \right\rceil = (q - 1) + 1 = q.$$

Hence the statement in this corollary is equivalent to Theorem 2. □

The following theorem is for the partition class of graphs in $\mathcal{G}_{w<\sigma}$. The proof is trivial. Note that for $0 < w < \sigma$, the term $\left\lceil \frac{w-\sigma}{\sigma} \right\rceil + 1 = 1$.

Theorem 3 *For the partition class* $\mathcal{G}_{w<\sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, w = Wt(G)$ *and* $w < \sigma\}$

$$\min_{G \in \mathcal{G}_{w<\sigma}}\{Wt(mwm(G))\} = 1.$$

3 Conclusion

In this paper, we have given a tight lower bound $\lceil \frac{w-\sigma}{\sigma} \rceil + 1$ for the weights of maximum weight matching of the bipartite graphs each having fixed weight as w and the size of each of the two non-empty subsets of the vertex partition as σ. The above finding may be applied in stringology, monitoring computer networks, computer vision, pattern recognition, compiler design with cloud architecture and telecommunication network to minimize the cost, time with the least traffic load and critical path routing.

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