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Modified Sombor Spectral Radii and Modified Sombor Energies of Splitting and Shadow Graphs

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Abstract

Gutman et al. introduced Sombor and Modified Somber index because of the rapidly growing applications in chemistry and network analysis. Graph energy $\varepsilon(G)$ and the spectral radius $\wp(G)$ of graph G are essential components that are associated with the eigenvalues of the matrix of graph G and, chemically, with the intermolecular forces. These graph invariants have many useful applications in computer sciences, networking, and molecular computing. There are numerous variants of the $\varepsilon(G)$ and $\wp(G)$ attained by substituting another matrix in place of an adjacency matrix. Modified Sombor energy, $MS\varepsilon(G)$ is defined as the sum of absolute eigenvalues of the modified Sombor matrix or in other words we can say that modified Sombor energy, $MS\varepsilon(G)$ is the trace norm of modified Sombor matrix. Modified Sombor spectral radius, $\wp MS$ is defined as the largest absolute eigenvalue of the modified Sombor matrix. The major focus of this article is on the $MS\varepsilon(G)$, $\wp MS$ of the generalized shadow and splitting graphs. The only realistic problem in which we are particularly interested in how $MS\varepsilon(Spl_t(G))$ and $MS\varepsilon(Sh_t(G))$ are comparable to $MS\varepsilon(G)$. On similar lines we are also interested in how $\wp MS(Spl_t(G))$ and $\wp MS(Sh_t(G))$ are comparable to $\wp MS(G)$. We were able to address these challenges by focusing on splitting and shadow graphs.

Keywords: MS spectral radius, splitting graph, MS energy, Shadow graph, eigenvalues.

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1 Introduction

QSAR/QSPR studies use physicochemical features and topological indices to predict the bioactivity of chemical substances. Topological indices link specific physicochemical aspects of chemical substances. Among degree-based topological indices, atom-bond connectivity (ABC) and geometric-arithmetic (GA) are crucial. These topological indices are used to link specific physicochemical features of chemical compounds, such as boiling point, stability, and strain energy. Authors in [14] computes the inverse sum indeg index for several graph operations like as lexicographic product, join, cartesian product, sequential join, and corona operation. The topological properties of neural networks can be explored using various graph theoretic parameters. The authors of [21] explore graph theoretic parameters and highlight some intriguing topological characteristics for several neural networks. The metric dimension and fault-tolerant metric dimension have potential uses in telecommunications, robot navigation, and geographic routing protocols. In [28] the authors investigate the fault-tolerant metric dimension of several interconnection networks. In [16], the authors investigate the metric and fault-tolerant dimension of specific families of interconnection networks. The authors of [19] investigate interconnection networks and obtain analytical closure conclusions for the general Randic index, as well as the ABC, first Zagreb, and GA indices. In [15], the authors computed the ABC index and the GA index for oxide and chain silicate networks. The authors of [32] demonstrate that the family of convex polytopes with an unbounded metric dimension has an unbounded fault-tolerant resolvability structure. Furthermore, they created three more infinite graph families that are closely linked to convex polytopes but have an unbounded metric dimension.

The relationship between graph structure and its eigenvalues has been established since the matrix-tree theorem was proven. The debate over the validity of the theory continues since graph invariants and spectral graph theory are related. Undoubtedly, the epidemic control model is one of the most well-known implementations. It is possible to determine the independence number and chromatic number using the eigenvalues. Spectral theory of graphs deeply looks at the relationship between a graph's topological properties and the spectral properties of its associated matrices. Initially, spectral theory of graphs investigated the eigenvalues of the adjacency matrix. Spectral theory of graphs aims to connect graph structures, such as connectedness, chromatic numbers, and diameters, to the spectra of corresponding matrices. Since the 1950s, spectral graph theory has been a focus of research in graph theory. Huckel first presented the HMO Theory in the 1930s. A molecular graph is used to illustrate conjugated hydrocarbons. The molecular graph's eigenvalues depict electron energy levels, jth molecular orbital energy is determined by $\varepsilon_j = \varrho + \lambda_j \sigma$ in HMO approximation. Total π-electron energy ε is calculated by adding the energies of all π-electrons available in the molecule. Gutman in [12] provided a general formula for calculating the total π-electron energy of a conjugated hydrocarbon using the HMO approach in the late 1970s and defined energy, $\varepsilon(G)$ as the sum of absolute eigenvalues of the adjacency matrix or in other words we can say that adjacency energy, $\varepsilon(G)$ is the trace norm of adjacency matrix. Graphs have significant invariants such as energy and spectral radius. Any regular graph G without isolated vertices will be taken into account here. A topological index is a single value that describes the properties of a molecular graph. Indices have remarkable use in chemistry, in the analysis of complex networks, proteomics and bioinformatics. These topics encompass a wide range of biological disciplines, including toxicology, microbiology, virology, and cancer research. The primary reason for indices popularity is their high flexibility in resolving seemingly unrelated challenges in all of these areas in a short amount of time. A wide variety of physiochemical properties of chemical compounds can be predicted by topological indices. The energy of a network is defined by a number of topological indices. Gutman established the modified Sombor matrix, which has entries that are $\frac{1}{\sqrt{d_i^2+d_j^2}}$ when $i \sim j$ and 0 elsewhere. $\varepsilon(G)$, was only pushed to be researched by a small group of scientists at first because it was such a bizarre idea. But mathematicians never really embraced the concept until the year 2000. Due of its widespread use and practical applicability across numerous industries, this concept is currently attracting a lot of attention. Bilal et al. investigated symmetric division deg spectral Radii and symmetric division deg energies [5], maximum and minimum degree spectral radii [38], atomic bond connectivity energies and atomic bond connectivity spectral radii [3], inverse sum indeg energies and inverse sum indeg spectral radii [6], and Randic and reciprocal randic spectral radii and energies [4] of the splitting and shadow graphs of any regular graphs. The authors of [18] obtained the relationship between other indices and the modified Sombor index, as well as certain bounds for the modified spectral radius and energy. Information and sources about spectral radii can be found in [7, 8, 36]. Horn et al. [17] and Gatmacher [10] both investigated matrix analysis. For more details and introductory ideas on graph energies, see [23]. [2, 27, 33] provide a history of the many uses of graph energy. Different variants of graph energies have different applications in different sectors of real life like in crystallography [1, 37], as well as in the theory of macromolecules [25, 34], biology [11], protein sequencing [9, 39, 35], air travel problems [20], and spacecraft architecture [26]. Sombor-like energies have some connections in chemistry and network-related applications. Some applications that forced us to study are the following: 1. By studying the Sombor-like energies of splitting graphs, we gain insights into the structural properties of networks like, Network Resilience: Higher Sombor-like energies energy implies greater robustness against vertex removal. Networks with higher splitting graph energy can withstand more vertex failures without significant loss of connectivity. Community Detection: Splitting graphs can be used in community detection algorithms. The energy of splitting graphs provides information about the community structure within a network. Graph Visualization: Splitting graphs can be visualized to reveal patterns and clusters within a network. " Spectral Properties: The eigenvalues of splitting graphs are related to the eigenvalues of the original graph, which has implications for spectral graph theory. 2. Shadow Graph Energies: Network Alignment: In biological networks or social networks, shadow graph energies can be used for network alignment tasks. Graph Clustering: Shadow graphs can reveal structural similarities between graphs, aiding in clustering and classification. Both of these operations help us in efficiently computing some properties as given in [40, 18].

2 Preliminaries

Fundamental concepts and background information pertinent to our key results are introduced here. The Sombor index, formulated by Ivan Gutman [13], and defined as $\sum_{ij}\sqrt{d_i^2+d_j^2}$, emerges from a geometric viewpoint of the degree radius, which measures the distance from the origin to the degree point (x, y). Redzepovic further explored the chemical significance of Sombor indices, as detailed in [29]. Additionally, Milovanovic and colleagues investigated the connections between Sombor indices and other indices [24]. Gutman also proposed a variant of the Sombor index, expressed as $\sum_{ij}\frac{1}{\sqrt{d_i^2+d_j^2}}$ [22]. Huang and associates established boundaries for both the modified Sombor spectral radius and modified Sombor energy [18]. Sombor index and its higher-order interactions have been explored in [31]. The authors of [40] calculate numerous constraints for modified Sombor spectral radius and modified Sombor energy. A matrix stores significantly more structural information than an index. The concept of the modified Sombor index requires that a square matrix MS be purposefully associated with the graph G. For further details and relevant references, please refer to [18]. According to [18], the modified Sombor MS matrix has entries given by

$$k_{ij} = \left\{ \begin{array}{ccc} \frac{1}{\sqrt{d_i^2+d_j^2}}, & when & v_i \sim v_j, \\ 0, & otherwise. & \end{array} \right\}$$

The modified Sombor matrix MS is a real symmetric matrix so all of its eigenvalues are real. If the modified Sombor matrix consist of the following eigenvalues $\gamma_1, \gamma_2, ..., \gamma_n$ with multiplicities $m_1, m_2, \cdots m_n$, respectively, then

$$specMS = \begin{pmatrix} \gamma_1 & \gamma_2 \cdots & \gamma_n \\ m_1 & m_2 \cdots & m_n \end{pmatrix}.$$

$$MS\varepsilon(G) = \sum_{i=1}^{n} |\gamma_i|,$$

where $\gamma_1, \gamma_2, ..., \gamma_n$ are the eigenvalues of modified Sombor matrix.

$$\rho MS(G) = \max_{i=1}^{n} |\gamma_i|.$$

In [30], authors introduced the splitting graph $Spl_1(G)$ of G and investigated its features. $Spl_t(G)$ is built for G by adding t more vertices for each vertex u, so that the newly added vertices are adjacent to each vertex in G that is adjacent to u. Figure 1 depicts a cycle graph with eight vertices, whereas Figure 2 illustrated the two splitting graph of C_8.

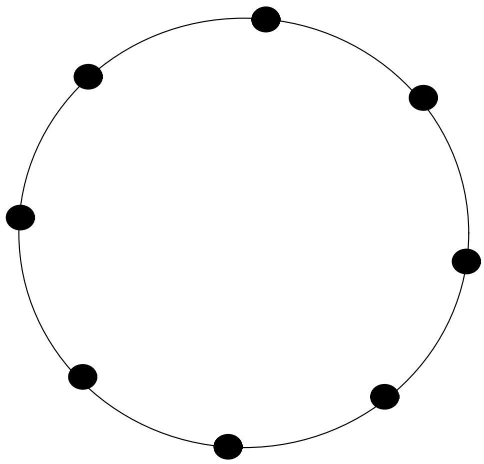

Figure 1: C_8.

$Sh_t(G)$ is created for a graph G by making t new copies of G and adding edges between the vertices of the copies if their corresponding vertices in G are adjacent. Figure 3 illustrated the three shadow graph of C_8.

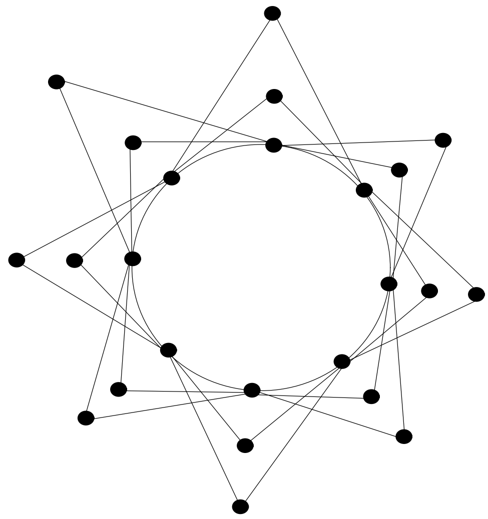

Figure 2: $Spl_2(C_8)$.

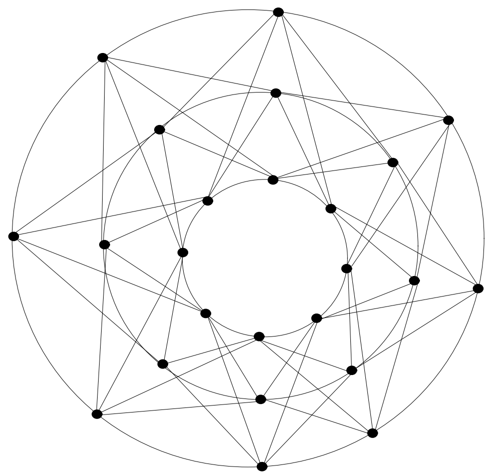

Figure 3: $Sh_3(C_8)$.

Let $L\epsilon Z^{s\times t}$, $M\epsilon Z^{u\times v}$.

$$L\bigotimes M = \begin{pmatrix} a_{11}M & . & . & .a_{1t}M \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{s1}M & . & . & .a_{st}M \end{pmatrix}.$$

Proposition 1.1 ([17]) *Let L has eigenvalue ι and M has eigenvalue κ, then $L \bigotimes M$ has an eigenvalue $\iota\kappa$.*

New findings are presented in the current article about modified Sombor energy and modified Sombor spectral radius. Section 3, deals with $MS\varepsilon(Spl_t(G))$ and $\rho MS(Spl_t(G))$. Section 4, deals with $MS\varepsilon(Sh_t(G))$ and $\rho MS(Sh_t(G))$.

3 Modified Sombor Energies and Modified Sombor Spectral Radii of $Spl_t(G)$

Graph operations are methods for creating new graphs from existing graphs. Splitting graph is a unary graph operation that create new graph from old one. When working with spectral parameters, it's important to understand the relationships between base graph and the newly created graph. We want to investigate how the modified Sombor energy and modified Sombor spectral radius of freshly generated graph ($Spl_t(G)$ depends on the modified Sombor energy and modified Sombor spectral radius of base graph G.

Theorem 1 *For $t \geq 1$, modified Sombor energy of $Spl_t(G)$ in terms of modified Sombor energy of original graph G is*

$$MS\varepsilon(Spl_t(G)) = \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}} MS\varepsilon(G).$$

Proof:

$$MS\left(Spl_t(G)\right) = \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_2 \\ \psi_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_2 & 0 & \cdots & 0 \end{bmatrix}.$$

Where $\psi_1 = \frac{1}{(t+1)} MS(G)$ and $\psi_2 = \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} MS(G)$.

$$MS\left(Spl_t(G)\right) = \begin{bmatrix} \frac{1}{(t+1)} MS(G) & \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} MS(G) & \dots & \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} MS(G) \\ \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} MS(G) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} MS(G) & 0 & \cdots & 0 \end{bmatrix}$$

$$= MS(G) \bigotimes \begin{bmatrix} \frac{1}{(t+1)} & \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} & \cdots & \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} \\ \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} & 0 & \cdots & 0 \end{bmatrix}$$

Let $[A] = c_{ij}$ having entries

$$c_{ij} = \begin{bmatrix} \frac{1}{(t+1)} & \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} & \cdots & \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} \\ \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\frac{t^2}{2}+t+1}} & 0 & \cdots & 0 \end{bmatrix}_{t+1}$$

To find $MS\varepsilon(Spl_t(G))$, it is essential to determine all the eigenvalues of the matrix $[A]$. We are currently in the process of calculating these eigenvalues. We denote required eigenvalues by α_1 and α_2. Evidently, we obtain

$$\alpha_1 + \alpha_2 = \frac{1}{(t+1)}. \tag{3.1}$$

Consider $[A^2] = d_{ij}$ having entries

$$d_{ij} = \begin{bmatrix} \frac{1}{(t+1)^2} + \frac{t}{(\frac{t^2}{2}+t+1} & \frac{1}{(t+1)(\sqrt{\frac{t^2}{2}+t+1})} & \cdots & \frac{1}{(t+1)(\sqrt{\frac{t^2}{2}+t+1})} \\ \frac{1}{(t+1)(\sqrt{\frac{t^2}{2}+t+1})} & \frac{1}{\frac{t^2}{2}+t+1} & \cdots & \frac{1}{\frac{t^2}{2}+t+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(t+1)(\sqrt{\frac{t^2}{2}+t+1})} & \frac{1}{\frac{t^2}{2}+t+1} & \cdots & \frac{1}{\frac{t^2}{2}+t+1} \end{bmatrix}_{t+1}$$

Then

$$\alpha_1^2 + \alpha_2^2 = tr(A^2) = \frac{1}{(t+1)^2} + \frac{2t}{(\frac{t^2}{2}+t+1)}. \tag{3.2}$$

After solving equations (3.1) and (3.2), the following outcomes are obtained

$$\alpha_1 = \frac{1}{2}(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}), \tag{3.3}$$

and

$$\alpha_2 = \frac{1}{2}(\frac{1}{t+1} - \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}). \tag{3.4}$$

$$Ch(A)=\alpha^{t-1}\left(\alpha-\tfrac{1}{2}(\tfrac{1}{t+1}+\sqrt{\tfrac{1}{(t+1)^2}+\tfrac{4t}{\frac{t^2}{2}+t+1}})\right)\left(\alpha-\tfrac{1}{2}(\tfrac{1}{t+1}-\sqrt{\tfrac{1}{(t+1)^2}+\tfrac{4t}{\frac{t^2}{2}+t+1}})\right)$$
$= 0.$

As a result, we obtained the spectrum $specA$,

$$specA = \begin{pmatrix} 0 & \frac{1}{2}(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}) & \frac{1}{2}(\frac{1}{t+1} - \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}) \\ t-1 & 1 & 1 \end{pmatrix}. \tag{3.5}$$

Since $MS(Spl_t(G)) = MS(G) \bigotimes A$. If $\varrho_1, \varrho_2, \varrho_3, ...\varrho_n$ are eigenvalues of $MS(G)$ then by Proposition 1.1

$$\begin{aligned} MS\varepsilon(Spl_t(G)) &= \sum_{i=1}^{n} \left| \frac{1}{2}(\frac{1}{t+1} \pm \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}})\gamma_i \right| \\ &= \sum_{i=1}^{n} |\gamma_i| \left[\frac{1}{2}(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}) \right] \\ &- \sum_{i=1}^{n} |\gamma_i| \left[\frac{1}{2}(\sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}} - \frac{1}{t+1}) \right] \\ &= \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}} MS\varepsilon(G). \end{aligned}$$

□

Proposition 3.1 *(i) Modified Sombor energy of $Spl_t(P(n,k))$ of Petersen graph $P(n,k)$, where $n \geq 2k+1$, is*

$$MS\varepsilon(Spl_t(P(n,k))) = \tfrac{1}{3\sqrt{2}} \sqrt{\tfrac{1}{(t+1)^2} + \tfrac{4t}{\frac{t^2}{2}+t+1}}$$

$$\cdot \left[\sum_{i=0}^{n-1} \left| \cos(\tfrac{2\pi\, i}{n}) + \cos(\tfrac{2\pi k\, i}{n}) \pm \sqrt{(\cos(\tfrac{2\pi\, i}{n}) - \cos(\tfrac{2\pi k\, i}{n}))^2 + 1} \right| \right]$$

(ii) Modified Sombor energy of $Spl_t(Y_n)$ of Prism graph Y_n is

$MS\varepsilon(Spl_t(Y_n)) = \frac{1}{3\sqrt{2}}\sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\left[\sum_{i=0}^{n-1}\left|2\cos\left(\frac{2\pi i}{n}\right) \pm 1\right|\right].$

(iii) Modified Sombor energy of $Spl_t(C_n)$ is

$$MS\varepsilon(Spl_t(C_n)) = \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\left[\sum_{i=0}^{n-1}\frac{1}{\sqrt{2}}(\cos\left(\frac{2\pi\ i}{n}\right))\right].$$

(iv) Modified Sombor energy of $Spl_t(K_n)$ is

$MS\varepsilon(Spl_t(K_n)) = \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}(\sqrt{2}).$

(v) Modified Sombor energy of $Spl_t(K_{n,n})$ is

$MS\varepsilon(Spl_t(K_{n,n})) = \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}(\sqrt{2}).$

Proof:

(i) By applying some simple algebra, we arrive at

$MS\varepsilon(P(n,k))$

$=\frac{1}{3\sqrt{2}}\sum_{i=0}^{n-1}\left|\cos\left(\frac{2\pi\ i}{n}\right) + \cos\left(\frac{2\pi k\ i}{n}\right) \pm \sqrt{(\cos\left(\frac{2\pi\ i}{n}\right) - \cos\left(\frac{2\pi k\ i}{n}\right)) + 1}\right|.$

Theorem 1 can be utilized to achieve the desired result, as the Petersen graph $P(n,k)$ is a regular graph.

(ii) By applying some simple algebra, we arrive at

$MS\varepsilon(Y_n) = \frac{1}{3\sqrt{2}}\sum_{i=0}^{n-1}\left|2\cos\left(\frac{2\pi i}{n}\right) \pm 1\right|.$

Theorem 1 can be utilized to achieve the desired result, as the prism graph Y_n is a regular graph.

(iii) Each vertex in C_n has a degree 2. So,

$$MS\left(C_n\right) = \begin{bmatrix} 0 & \frac{1}{\sqrt{8}} & 0 & 0 & \dots & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \frac{1}{\sqrt{8}} & 0 & 0 & \dots & \frac{1}{\sqrt{8}} & 0 \end{bmatrix}.$$

$$MSspec\,(C_n) = \frac{1}{\sqrt{2}}(\cos{(\frac{2\pi\ i}{n})})for i = 0, 1, 2, \cdots, n-1. \qquad (3.6)$$

$MS\varepsilon(C_n) = \sum_{i=0}^{n-1} \frac{1}{\sqrt{2}}(\cos{(\frac{2\pi\ i}{n})})$.

Theorem 1 can be employed to obtain the desired outcome.

(iv) Each vertex in K_n has a degree $n-1$. So,

$$MS\,(K_n) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & \cdots & \frac{1}{\sqrt{2(n-1)^2}} \\ \frac{1}{\sqrt{2(n-1)^2}} & 0 & \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & \cdots & \frac{1}{\sqrt{2(n-1)^2}} \\ \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & 0 & \frac{1}{\sqrt{2(n-1)^2}} & \cdots & \frac{1}{\sqrt{2(n-1)^2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & \frac{1}{\sqrt{2(n-1)^2}} & \cdots & 0 \end{bmatrix}.$$

$$MSspec\,(K_n) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2n-2} \\ 1 & n-1 \end{pmatrix}. \qquad (3.7)$$

$MS\varepsilon(K_n) = |\frac{1}{\sqrt{2}}| + (n-1)| - \frac{\sqrt{2}}{2n-2}|$.

$MS\varepsilon(K_n) = \frac{1}{\sqrt{2}} + (n-1)\frac{\sqrt{2}}{2n-2}$.

Finally, we have

$$MS\varepsilon(K_n) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}. \qquad (3.8)$$

We arrive at $MS\varepsilon(K_n) = \sqrt{2}$. Theorem 1 can be employed to obtain the desired outcome.

(v) Because $K_{n,n}$ is a n regular graph, so each vertex in it has a degree n.

$$MS\,(K_{n,n}) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2n^2}} & \frac{1}{\sqrt{2n^2}} & \cdots & \frac{1}{\sqrt{2n^2}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2n^2}} & \frac{1}{\sqrt{2n^2}} & \cdots & \frac{1}{\sqrt{2n^2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2n^2}} & \frac{1}{\sqrt{2n^2}} & \cdots & \frac{1}{\sqrt{2n^2}} \\ \hline \frac{1}{\sqrt{2n^2}} & \frac{1}{\sqrt{2n^2}} & \cdots & \frac{1}{\sqrt{2n^2}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2n^2}} & \frac{1}{\sqrt{2n^2}} & \cdots & \frac{1}{\sqrt{2n^2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2n^2}} & \frac{1}{\sqrt{2n^2}} & \cdots & \frac{1}{\sqrt{2n^2}} & 0 & 0 & \cdots & 0 \end{array}\right].$$

$$MSspec\,(K_{n,n}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 2n-2 & 1 \end{pmatrix}. \tag{3.9}$$

$MS\varepsilon(K_{n,n}) = |\frac{1}{\sqrt{2}}| + |-\frac{1}{\sqrt{2}}|.$

$MS\varepsilon(K_{n,n}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}.$

Finally, we have

$$MS\varepsilon(K_{n,n}) = \sqrt{2}. \tag{3.10}$$

Theorem 1 can be employed to obtain the desired outcome.

□

Theorem 2 *For $t \geq 1$, modified Sombor spectral radius of $Spl_t(G)$ in terms of modified Sombor spectral radius of original graph G is*

$$\wp MS(Spl_t(G)) = \wp MS(G)(\sqrt{\tfrac{1}{(t+1)^2} + \tfrac{4t}{\frac{t^2}{2}+t+1}}).$$

Proof: Using the spectrum $specA$,

$$specA = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right) & \frac{1}{2}\left(\frac{1}{t+1} - \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right) \\ t-1 & 1 & 1 \end{pmatrix}.$$

Since $MS(Spl_t(G)) = MS(G) \bigotimes A$. If $\varrho_1, \varrho_2, \varrho_3, ...\varrho_n$ are eigenvalues of $MS(G)$ then by Proposition 1.1

$$\begin{aligned} \wp MS(Spl_t(G)) &= max_{i=1}^{n} \left|(specA)\right) \gamma_i| \\ &= max_{i=1}^{n} |\gamma_i| \left[\frac{1}{2}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)\right] \\ &= \wp MS(G) \left[\frac{1}{2}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)\right]. \end{aligned}$$

□

Proposition 3.2 *(i) Modified Sombor spectral radius of $Spl_t(P(n,k))$ is*

$$\wp MS(Spl_t(P(n,k))) = \frac{1}{2\sqrt{2}} \left[\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)\right].$$

(ii) Modified Sombor spectral radius of $Spl_t(Y_n)$ *is*

$$\wp MS(Spl_t(Y_n)) = \frac{1}{2\sqrt{2}}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right).$$

(iii) Modified Sombor spectral radius of $Spl_t(C_n)$

$$\wp MS(Spl_t(C_n)) = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)\right].$$

(iv) Modified Sombor spectral radius of $Spl_t(K_n)$

$$\wp MS(Spl_t(K_n)) = \frac{1}{2}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)\left(\frac{1}{\sqrt{2}}\right).$$

(v) Modified Sombor spectral radius of $Spl_t(K_{n,n})$

$$\wp MS(Spl_t(K_{n,n})) = \frac{1}{2}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)\frac{1}{\sqrt{2}}.$$

Proof:

(i) Note that the spectrum of $P(n,k)$ is as follows: $spec[P(n,k)] =$

$$\begin{pmatrix} \frac{1}{3\sqrt{2}}\left(\cos\left(\frac{2\pi\, i}{n}\right) + \cos\left(\frac{2\pi k\, i}{n}\right) \pm \sqrt{(\cos\left(\frac{2\pi\, i}{n}\right) - \cos\left(\frac{2\pi k\, i}{n}\right)) + 1}\right) \\ n \end{pmatrix}.$$

Where $0 \le i \le n-1$

Since the range of $\cos x$ is $[-1,1]$, By applying some simple algebra, we arrive at $\wp MS(P(n,k)) = \frac{1}{\sqrt{2}}$. Applying Theorem 2, we obtain the result again.

(ii) The spectrum of Y_n is as follows:

$$spec(Y_n) = \begin{pmatrix} \frac{1}{3\sqrt{2}}\left(2\cos\left(\frac{2\pi i}{n}\right) + 1\right) & \frac{1}{3\sqrt{2}}\left(2\cos\left(\frac{2\pi i}{n}\right) - 1\right) \\ n & n \end{pmatrix}.$$

Where $0 \le i \le n-1$

We arrive at $\wp MS(Y_n) = \frac{1}{\sqrt{2}}$. By employing Theorem 2, we obtain the conclusion. This is due to the fact that the Prism graph Y_n is regular.

(iii) By utilizing the modified Sombor $MSspecC_n$
$MSspecC_n =$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}(\cos\frac{2\pi(0)}{n}) & \frac{1}{\sqrt{2}}(\cos\frac{2\pi(1)}{n}) & \frac{1}{\sqrt{2}}(\cos\frac{2\pi(2)}{n}) & \cdots \frac{1}{\sqrt{2}}(\cos\frac{2\pi(n-1)}{n}) \\ 1 & 1 & 1 & \cdots 1 \end{pmatrix}.$$

Since the range of $\cos x$ is $[-1,1]$, By applying some simple algebra, we arrive at $\wp MS(C_n) = \frac{1}{\sqrt{2}}$. Theorem 2 can be employed to obtain the desired outcome.

(iv) By utilizing the modified Sombor spectrum of K_n

$$MSspec\,(K_n) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2n-2} \\ 1 & n-1 \end{pmatrix}.$$

We arrive at $\wp MS(K_n) = \frac{1}{\sqrt{2}}$. Theorem 2 can be employed to obtain the desired outcome.

(v) By utilizing the modified Sombor spectrum of $K_{n,n}$

$$MSspec\,(K_{n,n}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 2n-2 & 1 \end{pmatrix}.$$

By applying some simple algebra, we arrive at $\wp MS(K_{n,n}) = \frac{1}{\sqrt{2}}$. Theorem 2 can be employed to obtain the desired outcome.

□

4 Modified Sombor Energies and Modified Sombor Spectral Radii of $Sh_t(G)$

Shadow graph is a unary graph operation that create new graph from old one. When working with spectral parameters, it's important to understand the relationships between base graph and the newly created graph. We want to investigate how the modified Sombor energy and modified Sombor spectral radius of freshly generated graph ($Sh_t(G)$ depends on the modified Sombor energy and modified Sombor spectral radius of base graph G.

Theorem 3 *For $t \geq 2$, modified Sombor energy of $Sh_t(G)$ in terms of modified Sombor energy of original graph G is*

$$MS\varepsilon(Sh_t(G)) = MS\varepsilon(G).$$

Proof:

$$MS\left(Sh_t(G)\right) = \begin{bmatrix} \psi_3 & \psi_3 & \dots & \psi_3 \\ \psi_3 & \psi_3 & \dots & \psi_3 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_3 & \psi_3 & \cdots & \psi_3 \end{bmatrix}.$$

$\psi_3 = (\frac{1}{t})MS(G)$.

$$MS\left(Sh_t(G)\right) = \begin{bmatrix} (\frac{1}{t})MS(G) & (\frac{1}{t})MS(G) & \dots & (\frac{1}{t})MS(G) \\ (\frac{1}{t})MS(G) & (\frac{1}{t})MS(G) & \dots & (\frac{1}{t})MS(G) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{t})MS(G) & (\frac{1}{t})MS(G) & \cdots & (\frac{1}{t})MS(G) \end{bmatrix}_t$$

$$= MS(G) \bigotimes \begin{bmatrix} \frac{1}{t} & \frac{1}{t} & \dots & \frac{1}{t} \\ \frac{1}{t} & \frac{1}{t} & \dots & \frac{1}{t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t} & \frac{1}{t} & \dots & \frac{1}{t} \end{bmatrix}_t$$

Let $[M] = m_{ij}$ having entries

$$m_{ij} = \begin{bmatrix} \frac{1}{t} & \frac{1}{t} & \dots & \frac{1}{t} \\ \frac{1}{t} & \frac{1}{t} & \dots & \frac{1}{t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t} & \frac{1}{t} & \dots & \frac{1}{t} \end{bmatrix}_t$$

To compute $MS\varepsilon(Sh_p(G))$, it is essential to obtain all the eigenvalues. Ultimately, $Ch(M) = \alpha^{t-1}(\alpha - 1) = 0$.

This equation highlights the relationship between the eigenvalues and the structure of the matrix M, providing critical insights into the underlying properties of the shadow graph. Accordingly, we obtain the spectrum as follows:

$$specM = \begin{pmatrix} 0 & 1 \\ t-1 & 1 \end{pmatrix}. \tag{4.1}$$

Since $MS(Spl_t(G)) = MS(G) \bigotimes M$. If $\varrho_1, \varrho_2, \varrho_3, ...\varrho_n$ are eigenvalues of $MS(G)$ then by Proposition 1.1

$$MS\varepsilon(Sh_t(G)) = MS\varepsilon(G).$$

□

Proposition 4.1 *(i) Modified Sombor energy of $Sh_t(Y_n)$is*

$$MS\varepsilon(Sh_t(Y_n)) = \frac{1}{3\sqrt{2}} \sum_{i=0}^{n-1} \left| 2\cos\left(\frac{2\pi i}{n}\right) \pm 1 \right|.$$

(ii) Modified Sombor energy of $Sh_t(C_n)$ is

$$MS\varepsilon(Sh_t(C_n)) = \sum_{i=0}^{n-1} \frac{1}{\sqrt{2}} (\cos\left(\frac{2\pi\ i}{n}\right)).$$

(iii) Modified Sombor energy of $Sh_t(K_n)$ is

$$MS\varepsilon(Sh_t(K_n)) = \sqrt{2}.$$

(iv) Modified Sombor energy of $Sh_t(K_{n,n})$ is

$$MS\varepsilon(Sh_t(K_{n,n})) = \sqrt{2}.$$

Proof:

(i) By applying some simple algebra, we arrive at

$$MS\varepsilon(Y_n) = \frac{1}{3\sqrt{2}} \sum_{i=0}^{n-1} \left| 2\cos\left(\frac{2\pi i}{n}\right) \pm 1 \right|.$$

We can conclude the proof using Theorem 3 as the Prism graph Y_n is regular.

(ii) By utilizing the modified Sombor spectrum of C_n

$$MSspec\,(C_n) = \frac{1}{\sqrt{2}} (\cos\left(\frac{2\pi\ i}{n}\right)) for i = 0, 1, 2, \cdots, n-1. \qquad (4.2)$$

$MS\varepsilon(C_n) = \sum_{i=0}^{n-1} \frac{1}{\sqrt{2}} (\cos\left(\frac{2\pi\ i}{n}\right))$.

Theorem 3 can be employed to obtain the desired outcome.

(iii) By utilizing modified Sombor spectrum of complete graph

$$MSspec\,(K_n) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2n-2} \\ 1 & n-1 \end{pmatrix}. \qquad (4.3)$$

$MS\varepsilon(K_n) = |\frac{1}{\sqrt{2}}| + (n-1)| - \frac{\sqrt{2}}{2n-2}|.$
$MS\varepsilon(K_n) = \frac{1}{\sqrt{2}} + (n-1)\frac{\sqrt{2}}{2n-2}.$
Finally, we have

$$MS\varepsilon(K_n) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}. \tag{4.4}$$

We arrive at $MS\varepsilon(K_n) = \sqrt{2}$. Theorem 3 can be employed to obtain the desired outcome.

(iv) By utilizing the modified Sombor spectrum of $K_{n,n}$

$$MSspec\,(K_{n,n}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 2n-2 & 1 \end{pmatrix}. \tag{4.5}$$

$MS\varepsilon(K_{n,n}) = |\frac{1}{\sqrt{2}}| + | - \frac{1}{\sqrt{2}}|.$
$MS\varepsilon(K_{n,n}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}.$
Finally, we have

$$MS\varepsilon(K_{n,n}) = \sqrt{2}. \tag{4.6}$$

Theorem 3 can be employed to obtain the desired outcome.

□

Theorem 4 *For $t \geq 2$, modified Sombor spectral radius of $Sh_t(G)$ in terms of modified Sombor spectral radius of original graph G is*

$$\wp MS(Sh_t(G)) = \wp MS(G).$$

This indicates that the MS spectral radius remains unchanged when transitioning from the base graph to its t-shadow graph, highlighting a significant structural property inherent in these graph transformations.

Proof: By applying the same justifications used for formula (5.1) in Theorem 3, we derive the following specification for the matrix M:

$$specM = \begin{pmatrix} 0 & 1 \\ t-1 & 1 \end{pmatrix}.$$

This formulation is significant, especially considering that the relationship between the modified Sombor matrices is given by $MS(Sh_t(G)) = MS(G) \bigotimes M$. Since $MS(Sh_t(G)) = MS(G) \bigotimes M$. If $\varrho_1, \varrho_2, \varrho_3, ...\varrho_n$ are eigenvalues of $MS(G)$. Then in view of Proposition 1.1, it is clear that

$$\wp MS(Sh_t(G)) = \wp MS(G).$$

□

Proposition 4.2 *(i) Modified Sombor spectral radius of $Sh_t(Y_n)$ is*

$$\wp MS(Sh_t(Y_n)) = \frac{1}{\sqrt{2}}.$$

(ii) Modified Sombor spectral radius of $Sh_t(C_n)$

$$\wp MS(Sh_t(C_n)) = \frac{1}{\sqrt{2}}.$$

(iii) Modified Sombor spectral radius of $Sh_t(K_n)$

$$\wp MS(Sh_t(K_n)) = \frac{1}{\sqrt{2}}.$$

(iv) Modified Sombor spectral radius of $Sh_t(K_{n,n})$

$$\wp MS(Sh_t(K_{n,n})) = \frac{1}{\sqrt{2}}.$$

Proof:

(i) By Utilizing the modified Sombor spectrum of Y_n , $\wp MS(Y_n) = \frac{1}{\sqrt{2}}$. We conclude the proof by using Theorem 4 as the Prism graph Y_n is regular.

(ii) By utilizing the modified Sombor

$MSspecC_n =$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}(\cos\frac{2\pi(0)}{n}) & \frac{1}{\sqrt{2}}(\cos\frac{2\pi(1)}{n}) & \frac{1}{\sqrt{2}}(\cos\frac{2\pi(2)}{n}) & \cdots \frac{1}{\sqrt{2}}(\cos\frac{2\pi(n-1)}{n}) \\ 1 & 1 & 1 & \cdots 1 \end{pmatrix}.$$

Since the range of $\cos x$ is $[-1, 1]$. Modified Sombor spectrum of C_n reveals that, $\wp MS(C_n) = \frac{1}{\sqrt{2}}$. We use Theorem 4 to conclude the proof.

(iii) By utilizing the modified Sombor spectrum of K_n

$$MSspec\,(K_n) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2n-2} \\ 1 & n-1 \end{pmatrix}.$$

Modified Sombor spectrum of K_n reveals that, $\wp MS(K_n) = \frac{1}{\sqrt{2}}$. We use Theorem 4 to conclude the proof.

(iv) By utilizing the modified Sombor spectrum of $K_{n,n}$

$$MSspec\,(K_{n,n}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 2n-2 & 1 \end{pmatrix}.$$

Modified Sombor spectrum of $K_{n,n}$ reveals that, $\wp MS(K_{n,n}) = \frac{1}{\sqrt{2}}$. We use Theorem 4 to conclude the proof.

□

5 Conclusion

Gutman et al. introduced Sombor and Modified Somber index because of the rapidly growing applications in chemistry and network analysis. Graph energy, $\varepsilon(G)$, and the spectral radius, $\wp(G)$, are two spectral parameters associated to a graph which exhibit structural properties of the graph G. Recently many variants of these two spectral parameters have been introduced. Modified Sombor energies and Modified Sombor spectral radius can be defined as the sum of absolute eigenvalues of the modified Sombor matrix, and the largest absolute eigenvalue of the modified Sombor matrix respectively.

One of the most contemporary theories within the realm of spectral graph theory is the concept of graph energy, which has been recently linked to the spectral radius. This innovative perspective highlights a significant intersection between mathematical theory and chemical applications, offering valuable insights into molecular structures and behaviors. The existing literature is abundant with research exploring these themes, underscoring their importance in both fields. Our investigation has centered on the spectral radii of larger graphs, a challenging yet essential endeavor. By examining the properties of splitting and shadow graphs, we have drawn some profound conclusions that contribute to the broader understanding of graph theory. Notably, our findings indicate that the spectral radii of these larger graphs can be expressed as multiples of the spectral radii of the base graph. This relationship not only reinforces the underlying structure of these graphs but also opens avenues for further exploration in related domains. Modified Sombor equienergetic graphs are two graphs having the same modified Sombor energy. Modified Sombor energy $MS\varepsilon(Spl_t(G)) = \sqrt{2}\sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}$, for every regular, complete multipartite graph, resulting in a wide range of modified Sombor equienergetic graphs. Modified Sombor spectral radius $\wp MS(Spl_t(G)) = \frac{1}{2\sqrt{2}}\left(\frac{1}{t+1} + \sqrt{\frac{1}{(t+1)^2} + \frac{4t}{\frac{t^2}{2}+t+1}}\right)$, for every regular graph. Modified Sombor energy $MS\varepsilon(Sh_t(G)) = \sqrt{2}$, for every regular, complete multipartite graph, resulting in a wide range of modified Sombor equienergetic graphs. Modified Sombor spectral radius $\wp MS(Sh_t(G)) = \frac{1}{\sqrt{2}}$, for every regular graph. Additionally, the implications of our research extend beyond theoretical boundaries. The insights gained from our analysis can significantly enhance our understanding of network robustness and the dynamics of virus transmission within networks. By leveraging the relationships established in this study, researchers can better model and predict behaviors in various applications, including epidemiology and network security. In summary, this research not only advances the theoretical foundations of spectral graph theory but also serves practical purposes in understanding complex systems. As we continue to explore these relationships, we anticipate that our findings will inspire further studies that bridge mathematics and real-world applications.

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