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z-ideals in the semiring $\mathcal{R}^+(L_\tau)$

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Abstract

Let $\mathcal{R}^+(L_\tau)$ denote the nonnegative τ-real-continuous function on a topoframe L_τ. We introduce the notion of z-ideals and maximal ideals in the semiring $\mathcal{R}^+(L_\tau)$ and state some results about them. Also, we show that the topoframe L_τ is a P-topoframe if and only if the semiring $\mathcal{R}^+(L_\tau)$ is a regular semiring.

Keywords: z-congruence, z-ideal, topoframe, τ-real-continuous function.

1 Introduction and Preliminaries

Pointfree topology (frame theory) focuses on the open sets rather than the points of a space. The concept of τ-real-continuous functions $\mathcal{R}(L_\tau)$ was first introduced by Estaji et al. [6]. In fact, they showed that $\mathcal{R}(L_\tau)$ is actually a generalization of $C(X)$, the f-ring of all continuous functions from the space X to the set $\mathbb{R}$. In [7], Estaji et al. introduced the notion of z-filters and z-ideals by using the concept of zero elements in modified pointfree topology and made ready some results about them. The semiring $C^+(X)$ is introduced and studied in [1] and [2]. Biswas and et al. in [3], presented a correlation between z-congruences on the ring $C(X)$ and z-congruences on the semiring $C^+(X)$ and provided a characterization of P-spaces by z-congruences on $C^+(X)$. Also, they studied various topologies on $C^+(X)$.

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In [5], Estaji et al. introduced the concepts of z-congruence in a ring $\mathcal{R}(L)$ and a semiring $\mathcal{R}^+(L)$ and gave a correlation between z-congruences on $\mathcal{R}(L)$ and z-congruences on $\mathcal{R}^+(L)$. Also, they showed that there is a one-one correspondence between z-ideals and z-congruences on a ring $\mathcal{R}(L)$ and a semiring $\mathcal{R}^+(L)$.

This paper is organized as follows. Section 1 presents the basic concepts and preliminaries, which will be used in the next sections. In Section 2, we introduce the concept of maximal ideals and z-ideals in a semiring $\mathcal{R}^+(L_\tau)$. In Section 3, we introduce the concepts of z-congruence in a ring $\mathcal{R}(L_\tau)$ and a semiring $\mathcal{R}^+(L_\tau)$. We give a correlation between z-congruences on $\mathcal{R}(L_\tau)$ and z-congruences on $\mathcal{R}^+(L_\tau)$. In Section 4, we investigate the equivalent conditions for a topoframe to become a P-topoframe.

A topoframe is a pair (L, τ), abbreviated L_τ, consisting of a frame $(L; \wedge, \vee, \bot, \top)$ and a subset τ of L satisfying the following conditions:

1. Every element p of τ has a complement p' in L.

2. The subset τ of L is a subframe of L.

The elements of L belonging to τ are called the open elements of L. An element in L is said to be closed if it is the complement of an open element. The set of all closed elements is denoted by $\tau' \colon = \{p' \colon p \in \tau\}$.

It is evident that $\big(\mathcal{P}(X), \mathcal{O}(X)\big)$ is a topoframe for every topological space X, where $\mathcal{O}(X)$ denotes the set of all open sets in X. We recall from [6] that a topoframe map f from a topoframe (L_1, τ_1) to a topoframe (L_2, τ_2) is a frame map f from L_1 to L_2 with the property $f(\tau_1) \subseteq \tau_2$ and also, a topoframe map f from the topoframe $\big(\mathcal{P}(\mathbb{R}), \mathcal{O}(\mathbb{R})\big)$ to a topoframe (L, τ) is called a τ-real-continuous function on L (or a real continuous function on L_τ). The set of all real-continuous functions on L_τ is denoted by $\mathcal{R}(L_\tau)$. For every $f, g \in \mathcal{R}(L_\tau)$ and every $\diamond \in \{+, \cdot, \vee, \wedge\}$, $f \diamond g \colon \mathcal{P}(\mathbb{R}) \to L$ is given by

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) \colon Y \diamond Z \subseteq X\},$$

where $Y \diamond Z = \{y \diamond z | y \in Y, z \in Z\}$. It is shown in [6] that $\mathcal{R}(L_\tau)$ is an f-ring with the indicated binary operations.

For every $f \in \mathcal{R}(L_\tau)$, $f(\{0\})$ is called a **zero-element** of f and is denoted by $z(f)$, and also, an element a in L is a zero-element of L_τ if $a = z(f)$ for some $f \in \mathcal{R}(L_\tau)$. Thus, z is a mapping from the ring $\mathcal{R}(L_\tau)$ onto the set of all zero-elements in L. For $A \subseteq \mathcal{R}(L_\tau)$, we write $z[A]$ to designate the family of zero-elements $\{z(f) \colon f \in A\}$. This is consistent with our notational convention for the image of a set under a mapping. On the other hand, the family $z[\mathcal{R}(L_\tau)]$ of all zero-elements in L will also be denoted, for simplicity, by $z[L_\tau]$. Also, a **cozero-element** of L_τ is defined by $coz(f) := f(-\infty, 0) \vee f(0, +\infty)$ for some $f \in \mathcal{R}(L_\tau)$. Obviously, $z(f) = \big(coz(f)\big)'$ (for more information on this ring, see [6]). Now, we recall some properties of L_τ, which will be used in what follows. For every $f, g \in \mathcal{R}(L_\tau)$ and every $c \in \mathbb{R}$, we have

1. $z(f+g) = z(f) \wedge z(g)$, while $f, g \geq \mathbf{0}$;
2. if $\mathbf{0} \leq f \leq g$, then $z(f) \geq z(g)$;
3. for every $n \in \mathbb{N}$, $z(f) = z(-f) = z(|f|) = z(f^n)$;
4. $z(fg) = z(f) \vee z(g)$;
5. $z(f+g) \geq z(f) \wedge z(g)$;
6. $z(f) \wedge z(g) = z(|f| + |g|) = z(f^2 + g^2)$;
7. $z(\mathbf{1}) = \perp$. Moreover, $z(f) = \top$ if and only if $f = \mathbf{0}$;
8. $z(f - \mathbf{c}) = f(c)$;
9. $z((f - \mathbf{c})^+) = f(-\infty, c]$ and $z((f - \mathbf{c})^-) = z((\mathbf{c} - f)^+) = f[c, +\infty)$;
10. $z(f^+) = f(-\infty, 0]$ and $z(f^-) = f[0, +\infty)$.

We recall the notion of a z-ideal of a ring A as was introduced by Mason in [12]. In the lattice theory this notion is known as "z-ideals à la Mason". Denote by $Max(A)$ the set of all maximal ideals of a ring A. For $a \in A$ and $S \subseteq A$, let

$$\mathcal{M}(a) = \{M \in Max(A)\colon a \in M\} \text{ and } \mathcal{M}(S) = \{M \in Max(A)\colon S \subseteq M\}.$$

Note that, since an ideal contains an element if and only if it contains the principal ideal generated by the element, we have that $\mathcal{M}(a) = \mathcal{M}(\langle a \rangle)$. An ideal I of a ring A is called a z-ideal à la Mason if whenever $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ and $a \in I$, then $b \in I$. Estaji et al. in [7] define a z-ideal of $\mathcal{R}(L_\tau)$ "topologically". An ideal I of a ring $\mathcal{R}(L_\tau)$ is called a **z-ideal** if the condition $z(f) \leq z(g)$, for $f \in I$ and $g \in \mathcal{R}(L_\tau)$, implies $g \in I$.

It is evident that $z[L_\tau]$ is a sublattice of L, and a proper filter of $z[L_\tau]$ is called a **z-filter** on L_τ. Therefore, if $\mathcal{F}$ is a z-filter on L_τ, then

(1) $\bot \notin \mathcal{F} \subseteq z[L_\tau]$ and $\top \in \mathcal{F}$,

(2) for every $a, b \in \mathcal{F}$, there exists $\bot \neq c \in \mathcal{F}$ that $c \leq a \wedge b$, and

(3) if $b \in \mathcal{F}$, $a \in z[L_\tau]$, and $b \leq a$, then $a \in \mathcal{F}$.

Also, the set of all z-filters on L_τ will be denoted by $\mathrm{zFil}(L_\tau)$.

Proposition 1 ([7]) *Let I be a proper ideal in $\mathcal{R}(L_\tau)$. Then the family $z[I] = \{z(f) \colon f \in I\}$ is a z-filter on L_τ.*

We recall from [9, 15] that a **semiring** is a nonempty set S on which the addition and multiplication operations have been defined such that the following conditions are satisfied:

(1) $(S, +)$ is a commutative monoid with identity element 0.

(2) $(S, \cdot)$ is a commutative monoid with identity element 1_R.

(3) Multiplication distributes over addition from either side.

(4) $0r = 0 = r0$ for all $r \in S$.

An element r of a semiring S is a unit if and only if there exists an element r' of S satisfying $rr' = 1 = r'r$. We denote the set of all units of S by $U(S)$. A semiring S is said to be **positive** if for each $x \in S$, $1+x \in U(S)$. A nonempty subset I of S is called an **ideal** of S if $a+b \in I$ and $ra \in I$ for all $a, b \in I$ and $r \in S$. An ideal I of S is said to be proper if $I \neq S$. Moreover, S and $\{0\}$ are said to be trivial ideals of S. Denote by $\mathcal{ID}(S)$ the family of all ideals of S. For an ideal I of S, the set $\overline{I} = \{x \in S \colon x + a = b \text{ for some } a, b \in I\}$ is called the **subtractive closure** or k**-closure** of I in S. Then $\overline{I}$ is an ideal of S, and it holds that $I \subseteq \overline{I}$ and $\overline{\overline{I}} = I$. An ideal I of S is called a **subtractive ideal** or k**-ideal** of S if $\overline{I} = I$. Denote by $\mathcal{KI}(S)$ the family of all k-ideals of S. Also, a proper ideal M of a semiring S is called a **maximal ideal** of S if $M \subseteq I \subseteq S$ for any ideal I of S implies either $I = M$ or $I = S$. We denote the set of all maximal ideals of S by $Max(S)$.

A **congruence** relation on a semiring S is an equivalence relation ρ on S that $(a, b) \in \rho$ implies $(a + x, b + x) \in \rho$ and $(ax, bx) \in \rho$ for every $a, b, x \in S$. The family of all congruences on S is denoted by Cong(S). The set Cong(S) with respect to the inclusion generates an algebraic lattice: $\rho \subseteq \tau$ means that $a\rho b$ implies $a\tau b$ for all $a, b \in S$. Congruence relation ρ on S is called **cancellative** if $(a + x, b + x) \in \rho$ implies $(a, b) \in \rho$ for every

$a, b, x \in S$ [9]. Also, a congruence ρ on S is called a **subtractive congruence** or k**-congruence** if there is an ideal I of S such that $\rho = k_I$, where

$$k_I := \big\{(f, g)\colon f, g \in S \text{ and } f + h = g + k \text{ for some } h, k \in I\big\}.$$

Denote by $KC(S)$ the family of all k-congruences on a semiring S (see [10]).

Now, let

$$\mathcal{R}^+(L_\tau) := \big\{\alpha \in \mathcal{R}(L_\tau)\colon \alpha \geq \mathbf{0}\big\}$$

be the positive cone of the ring $\mathcal{R}(L_\tau)$. It is easy to see that $\mathcal{R}^+(L_\tau)$ is a commutative semiring.

2 Maximal Ideals and z-ideals in a Semiring $\mathcal{R}^+(L_\tau)$

We recall that if B is a subring of a ring A and I is an ideal of A, then $I \cap B$ is an ideal of B called the contraction of I and denoted by I^c. In this section, we discuss the contraction of maximal ideals in a ring $\mathcal{R}(L_\tau)$ and a semiring $\mathcal{R}^+(L_\tau)$. Also, we define the concept of z-ideal in the semiring $\mathcal{R}^+(L_\tau)$.

It is clear that in any f-ring, the inverse of a positive invertible element is positive.

Proposition 2 *Let M be a maximal ideal of a ring $\mathcal{R}(L_\tau)$. Then $M^c = M \cap \mathcal{R}^+(L_\tau)$ is a maximal ideal of the semiring $\mathcal{R}^+(L_\tau)$.*

Proof: It is clear that M^c is an ideal of semiring $\mathcal{R}^+(L_\tau)$. Now, let $\alpha \in \mathcal{R}^+(L_\tau) \setminus M^c$. Then $\alpha^2 \geq \mathbf{0}$, and so $\alpha^2 \in \mathcal{R}^+(L_\tau) \setminus M^c$. Hence $\alpha^2 \notin M$, and so $M + (\alpha^2) = \mathcal{R}(L_\tau)$. Then there exist $\gamma \in M$ and $\beta \in \mathcal{R}(L_\tau)$ such that $\beta\alpha^2 + \gamma = \mathbf{1}$. Hence

$$\top = coz(\mathbf{1}) = coz(\beta\alpha^2 + \gamma) \leq coz(\beta\alpha^2) \vee coz(\gamma) = coz(\beta^2\alpha^4 + \gamma^2),$$

which means that $\beta^2\alpha^4 + \gamma^2$ is invertible. Then, there exists $u \in U(\mathcal{R}^+(L_\tau))$ such that $u(\beta^2\alpha^4 + \gamma^2) = \mathbf{1}$. Therefore M^c is an maximal ideal of the semiring $\mathcal{R}^+(L_\tau)$. $\square$

Remark 3 *Every z-filter on L_τ is contained in a z-ultrafilter on L_τ.*

In the following proposition, we characterize the maximal ideal of semiring $\mathcal{R}^+(L_\tau)$.

Proposition 4 *Let M be a maximal ideal of a semiring $\mathcal{R}^+(L_\tau)$. Then there exists z-ultrafilter $\mathcal{F}$ of L_τ such that $M = z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau)$.*

Proof: Let $z[M] = \{z(\alpha)\colon \alpha \in M\}$. First, we show that $z[M]$ is a z-filter on L_τ. Since M contains no unit, we conclude from [7, Theorem 3.8] that $\perp \notin z[M]$. Let $z_1, z_2 \in z[M]$. Then there exist $\alpha, \beta \in M$ such that $z_1 = z(\alpha)$ and $z_2 = z(\beta)$. Since M is an ideal, $\alpha + \beta \in M$. Hence

$$z_1 \wedge z_2 = z(\alpha) \wedge z(\beta) = z(\alpha + \beta) \in z[M].$$

Now, let $z \in z[M]$ and let $z_1 \in z[L_\tau]$ with $z \leq z_1$. Then there exist $\alpha \in M$ and $\beta \in \mathcal{R}(L_\tau)$ such that $z = z(\alpha)$ and $z_1 = z(\beta)$. Since M is an ideal, we have $\alpha\beta^2 \in M$. Hence if $z \leq z_1$, then $z_1 = z \vee z_1 = z(\alpha) \vee z(\beta) = z(\alpha) \vee z(\beta^2) = z(\alpha\beta^2) \in z[M]$. Then $z[M]$ is a z-filter on L_τ. Hence by Remark 3, $z[M]$ is contained in a z-ultrafilter $\mathcal{F}$ on L_τ. By [7, Proposition 5.3], $z^{-1}(\mathcal{F})$ is a maximal ideal in $\mathcal{R}(L_\tau)$. Therefore, by Proposition 2, $z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau)$ is a maximal ideal of semiring $\mathcal{R}^+(L_\tau)$ and $M \subseteq z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau)$. Since M is a maximal ideal, we conclude that $M = z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau)$. □

Proposition 5 *If M is a maximal ideal of semiring $\mathcal{R}^+(L_\tau)$, then*

$$M = \{\beta \in \mathcal{R}^+(L_\tau)\colon z(\alpha) \leq z(\beta) \, \textit{for some } \alpha \in M\}.$$

Proof: Let M be a maximal ideal of semiring $\mathcal{R}^+(L_\tau)$. By Proposition 4, there exists a z-ultrafilter $\mathcal{F}$ of L_τ such that $M = z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau)$. Let $(\alpha, \beta) \in M \times \mathcal{R}^+(L_\tau)$ with $z(\alpha) \leq z(\beta)$ be given. Then $z(\alpha) \in \mathcal{F}$, which implies that $z(\beta) \in \mathcal{F}$, and this entails that $\beta \in z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau) = M$. □

Let S be a semiring, let $a \in S$, and let M_a^+ be the intersection of all maximal ideals containing a. If S is a positive semiring, then by [13, Theorem 2],

$$M_a^+ = \big\{x \in S\colon \text{ for all } y \in S, a + y \notin U(S) \Rightarrow a + x + y \notin U(S)\big\}.$$

Now, let $\alpha \in \mathcal{R}^+(L_\tau)$. By [7, Theorem 3.8], a frame map α is invertible if and only if $z(\alpha) = \perp$. Then for every $\alpha \in \mathcal{R}^+(L_\tau)$,

$$M_\alpha^+ = \{\beta \in \mathcal{R}^+(L_\tau)\colon \text{ for all } \gamma \in \mathcal{R}^+(L_\tau), z(\alpha+\gamma) \neq \perp \Rightarrow z(\alpha+\beta+\gamma) \neq \perp\}.$$

In the following proposition, we bring some properties of M_α^+.

Proposition 6 *Let $\alpha, \beta \in \mathcal{R}^+(L_\tau)$ be given. Then, the following statements are true.*

(1) *If $z(\alpha\beta) = z(\beta)$, then $M_{\alpha\beta}^+ = M_\beta^+$.*

(2) *If* $\alpha \in M_\alpha^+$ *with* $z(\alpha) \leq z(\beta)$*, then* $\beta \in M_\alpha^+$*.*

(3) *If* M_α^+ *is a proper ideal of semiring* $\mathcal{R}^+(L_\tau)$*, then* $z(\alpha) \wedge z(\beta) \neq \bot$ *for every* $\beta \in M_\alpha^+$*.*

Proof: (1). Let $\gamma \in M_{\alpha\beta}^+$ and let $z(\beta + \delta) \neq \bot$ for $\delta \in \mathcal{R}^+(L_\tau)$. Then

$$\begin{aligned} z(\alpha\beta + \delta) &= z(\alpha\beta) \wedge z(\delta) \\ &= z(\beta) \wedge z(\delta) \\ &= z(\beta + \delta), \end{aligned}$$

and so $z(\alpha\beta + \delta) \neq \bot$. Since $\gamma \in M_{\alpha\beta}^+$, we have $z(\alpha\beta + \delta + \gamma) \neq \bot$. Then

$$\begin{aligned} z(\beta + \gamma + \delta) &= z(\beta) \wedge z(\gamma) \wedge z(\delta) \\ &= z(\alpha\beta) \wedge z(\gamma) \wedge z(\delta) \\ &= z(\alpha\beta + \gamma + \delta) \\ &\neq \bot. \end{aligned}$$

Hence, $\gamma \in M_\beta^+$ and so $M_{\alpha\beta}^+ \subseteq M_\beta^+$. The proof of the converse is similar to the above argument.

(2) By defintion of M_α^+ and Proposition 5, the proof is straightforward.

(3). Let $\beta \in M_\alpha^+$ and $z(\alpha) \wedge z(\beta) = \bot$. If α is invertible, then $z(\alpha) = \bot$ and $M_\alpha^+ = \mathcal{R}^+(L_\tau)$, which is a contradiction. Then $z(\alpha) \neq \bot$. Hence there exists z-ultrafilter $\mathcal{F}$ of L_τ, such that $z(\alpha) \in \mathcal{F}$. Then, by Proposition 2, $M := z^{-1}(\mathcal{F}) \cap \mathcal{R}^+(L_\tau)$ is a maximal ideal of $\mathcal{R}^+(L_\tau)$ and $M_\alpha^+ \subseteq M$, which implies that $z(\alpha) \wedge z(\beta) \in \mathcal{F}$, which means that $z(\alpha) \wedge z(\beta) \neq \bot$. □

Definition 7 *The proper ideal* I *of a semiring* $\mathcal{R}^+(L_\tau)$ *is called a* z*-ideal if whenever* $\alpha \in I$*, then* $M_\alpha^+ \subseteq I$*.*

It is evident that every maximal ideal of semiring $\mathcal{R}^+(L_\tau)$ is a z-ideal and the intersection of an arbitrary (nonempty) family of z-ideals is a z-ideal. Then M_α^+ is also a z-ideal.

Proposition 8 *Let* I *be a* z*-ideal of a semiring* $\mathcal{R}^+(L_\tau)$*. Then, for any* $\alpha, \beta \in \mathcal{R}^+(L_\tau)$*,* $\alpha \in I$ *and* $z(\alpha) \leq z(\beta)$ *imply* $\beta \in I$*.*

Proof: It is clear that by Proposition 6. □

3 On z-congruences on $\mathcal{R}(L_\tau)$ and $\mathcal{R}^+(L_\tau)$

In this section, we introduce the concept of z-congruence on a ring $\mathcal{R}(L_\tau)$ and a semiring $\mathcal{R}^+(L_\tau)$ and give some properties of them. Also, we examine the equivalence conditions for the z-congruences of a ring $\mathcal{R}(L_\tau)$ and a semiring $\mathcal{R}^+(L_\tau)$. Moreover, we show that there is a bijection between the collection of all z-congruences and the collection of z-ideals in a ring $\mathcal{R}(L_\tau)$ and a semiring $\mathcal{R}^+(L_\tau)$.

We recall from [2] that, for any $f, g \in C^+(X)$, $E(f,g) = \{x \in X : f(x) = g(x)\}$ is said to be the agreement set of f and g. $E(f,g)$ is cleary a zero-set in X and each zero set in X is an agreement set $E(f,g)$ for some $f, g \in C^+(X)$. It is further noted in [3], that for $f, g \in C^+(X)$, $E(f,g) = Z(f-g) = Z(|f-g|)$ where $|f-g| \in C^+(X)$. Also, $E(\rho) := \big\{E(f,g) \text{ and } (f,g) \in \rho\big\}$ for any congruence relation ρ on $C^+(X)$.

Now, let ρ be a congruence relation on $\mathcal{R}(L_\tau)$ ($\mathcal{R}^+(L_\tau)$). Then we set

$$E(\rho) := \big\{z(f-g) \colon f, g \in \mathcal{R}(L_\tau) \text{ and } (f,g) \in \rho\big\},$$

and

$$E^+(\rho) := \big\{z(f-g) \colon f, g \in \mathcal{R}^+(L_\tau) \text{ and } (f,g) \in \rho\big\}.$$

Definition 9 *We call a proper congruence ρ on $\mathcal{R}(L_\tau)$ (or on $\mathcal{R}^+(L_\tau)$) a **z-congruence** if $z(f-g) \in E(\rho)$ (or $z(f-g) \in E^+(\rho)$) implies that $(f,g) \in \rho$ for every $f, g \in \mathcal{R}(L_\tau)$ (or $f, g \in \mathcal{R}^+(L_\tau)$). Also, the set of all z-congruences on $\mathcal{R}(L_\tau)$ (or on $\mathcal{R}^+(L_\tau)$) will be denoted by* $\mathrm{zCong}(L_\tau)$ *(or* $\mathrm{zCong}^+(L_\tau)$*).*

The concept of z-filters in L_τ was introduced by using the concept of zero element in [7]. Also, it was investigated the relationship between ideals of a ring $\mathcal{R}(L_\tau)$ and z-filters on L_τ. In the following proposition, we show that there is a correspondence between proper congruences on a ring $\mathcal{R}(L_\tau)$ and z-filters on L_τ. For every subset I of $\mathcal{R}(L_\tau)$, set

$$\rho_I := \big\{(f,g) \colon f, g \in \mathcal{R}(L_\tau) \text{ and } f - g \in I\big\}.$$

Now, let

$$\rho_I^+ := \big\{(f,g) \colon f, g \in \mathcal{R}^+(L_\tau) \text{ and } f - g \in I\big\}$$

for every subset I of $\mathcal{R}^+(L_\tau)$.

Proposition 10 *Let L_τ be a topoframe. Then, the following statements are true:*

(1) *For an ideal I of $\mathcal{R}(L_\tau)$ (or $\mathcal{R}^+(L_\tau)$), $E(\rho_I) = z[I]$ (or $E^+(\rho_I^+) = z[I]$).*

(2) *If ρ is a proper congruence relation on $\mathcal{R}(L_\tau)$ (or ρ is a proper cancellative congruence relation on $\mathcal{R}^+(L_\tau)$), then $E(\rho)$ (or $E^+(\rho)$) is a z-filter on L_τ.*

(3) *If $\mathcal{F}$ is a z-filter on L_τ, then $E^{-1}(\mathcal{F}) := \big\{(f,g) \in \mathcal{R}(L_\tau)^2 \colon z(f-g) \in \mathcal{F}\big\}$ is a proper congruence on $\mathcal{R}(L_\tau)$.*

Proof:

(1) Let I be an ideal of $\mathcal{R}(L_\tau)$. If $f \in I$, then $(f, \mathbf{0}) \in \rho_I$. Thus $z(f) \in E(\rho_I)$. If $z \in E(\rho_I)$, then there exists an element $(f,g) \in \rho_I$ such that $z = z(f-g) \in z[I]$. Hence, $z[I] = E(\rho_I)$.

(2) If $\bot \in E(\rho)$, then there exists an element $(f,g) \in \rho$ such that $z(f-g) = \bot$, which implies that $f-g$ is a unit of $\mathcal{R}(L_\tau)$. Then

$$\begin{aligned}(f,g) \in \rho &\Rightarrow (f-g, \mathbf{0}) \in \rho \Rightarrow \big((f-g)(f-g)^{-1}, \mathbf{0}\big) \in \rho \\ &\Rightarrow (\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{1}) \in \rho \Rightarrow \rho = \mathcal{R}(L_\tau) \times \mathcal{R}(L_\tau),\end{aligned}$$

which is a contradiction. Hence, $\bot \notin E(\rho)$. Let $z_1, z_2 \in E(\rho)$ be given. Then there exist $(f_1, g_1), (f_2, g_2) \in \rho$ such that $z_1 = z(f_1 - g_1)$ and $z_2 = z(f_2 - g_2)$, which implies that

$$\begin{aligned}z_1 \wedge z_2 &= z(f_1 - g_1) \wedge z(f_2 - g_2) \\ &= z(f_1^2 + f_2^2 + g_1^2 + f_2^2 - 2f_1g_1 - 2f_2g_2)\end{aligned}$$

and $(f_1^2 + f_2^2 + g_1^2 + g_2^2, 2f_1g_1 + 2f_2g_2) \in \rho$. Hence, $z_1 \wedge z_2 \in E(\rho)$. Now, let $(z_1, z_2) \in E(\rho) \times z[L_\tau]$ with $z_1 \leq z_2$ be given. Then there exists an element $\big((f_1, g_1), g\big) \in \rho \times \mathcal{R}(L_\tau)$ such that $z_1 = z(f_1 - g_1)$ and $z_2 = z(g)$. From $z_2 = z_1 \vee z_2 = z\big(g(f_1 - g_1)\big)$ and $(gf_1, gg_1) \in \rho$, we conclude that $z_2 \in E(\rho)$. Therefore, $E(\rho)$ is a z-filter on L_τ.

(2) Let $(f,g), (g,h) \in E^{-1}(\mathcal{F})$ be given. Then $z(f-g), z(g-h) \in \mathcal{F}$, which implies that $z(f-h) = z(f-g+g-h) \geq z(f-g) \wedge z(g-h) \in \mathcal{F}$, and so $(f,h) \in E^{-1}(\mathcal{F})$. Hence, $E^{-1}(\mathcal{F})$ is an equivalence relation on

$\mathcal{R}(L_\tau)$ and $E^{-1}(\mathcal{F}) \neq \mathcal{R}(L_\tau)^2$. Let $(f,g) \in E^{-1}(\mathcal{F})$ and $h \in \mathcal{R}(L_\tau)$ be given. Then

$$\begin{aligned} z(f-g) \in \mathcal{F} &\Rightarrow z\big(f+h-(g+h)\big) \in \mathcal{F} \text{ and } z\big(h(f-g)\big) \geq z(f-g) \\ &\Rightarrow (f+h, g+h) \in E^{-1}(\mathcal{F}) \text{ and } z\big(h(f-g)\big) \in \mathcal{F} \\ &\Rightarrow (f+h, g+h), (fh, gh) \in E^{-1}(\mathcal{F}). \end{aligned}$$

Therefore, $E^{-1}(\mathcal{F})$ is a proper congruence on $\mathcal{R}(L_\tau)$. □

In an arbitrary ring R, there is a bijection between the collection of all congruences on R and the collection of ideals of R (see [11, Remark 7.6]). For every binary relation ρ on $\mathcal{R}(L_\tau)$, let

$$I_\rho := \big\{f-g \colon f,g \in \mathcal{R}(L_\tau) \text{ and } (f,g) \in \rho\big\}.$$

In the following proposition, we show that there is a bijection between the collection of z-congruences on $\mathcal{R}(L_\tau)$ and the collection of z-ideals of $\mathcal{R}(L_\tau)$.

Proposition 11 *Let L_τ be a topoframe. Then, the following statements are true:*

(1) *If ρ is a proper congruence on $\mathcal{R}(L_\tau)$, then I_ρ is a proper ideal of $\mathcal{R}(L_\tau)$ and $\rho = \rho_{I_\rho}$. In particular, if ρ is a z-congruence on $\mathcal{R}(L_\tau)$, then I_ρ is a z-ideal of $\mathcal{R}(L_\tau)$.*

(2) *If I is a proper ideal of $\mathcal{R}(L_\tau)$, then ρ_I is a proper congruence on $\mathcal{R}(L_\tau)$ and $I = I_{\rho_I}$. In particular, if I is a z-ideal of $\mathcal{R}(L_\tau)$, then ρ_I is a z-congruence on $\mathcal{R}(L_\tau)$.*

Proof:

(1) Let $v, w \in I_\rho$ be given. Then there exist $(f,g), (h,k) \in \rho$ such that $v = f-g$ and $w = h-k$, which implies that $(f+k, g+h) \in \rho$ and $v-w = (f+k)-(g+h)$; that is, $v-w \in I_\rho$. Now, assume that $(v,w) \in I_\rho \times \mathcal{R}(L_\tau)$. Thus, there exists an element (f,g) in ρ such that $v = f-g$, which implies that $vw = fw - gw$ and $(fw, gw) \in \rho$, and so, $vw \in I_\rho$. Therefore, I_ρ is an ideal of $\mathcal{R}(L_\tau)$. Also, from

$$(f,g) \in \rho_{I_\rho} \Leftrightarrow f-g \in I_\rho \Leftrightarrow (f,g) \in \rho,$$

we infer that $\rho = \rho_{I_\rho}$.

Now, let $(f, g) \in I_\rho \times \mathcal{R}(L_\tau)$ with $z(f) \leq z(g)$ be given. Then there exists an element $(h, k) \in \rho$ such that $f = h - k$, which implies that $z(h-k) = z(f) \leq z(g) = z(g - \mathbf{0})$. By Proposition 10, $z(g - \mathbf{0}) \in E(\rho)$ and so $(g, \mathbf{0}) \in \rho$. Thus $g = g - \mathbf{0} \in I_\rho$. It is evident that I_ρ is a proper ideal of $\mathcal{R}(L_\tau)$. Then I_ρ is a z-ideal of $\mathcal{R}(L_\tau)$.

(2) It is evident that ρ_I is an equivalence relation on $\mathcal{R}(L_\tau)$. Let $(f, g), (h, k) \in \rho_I$ be given. Then $f - g, h - k \in I$, which implies that $(f+h)-(g+k) \in I$ and

$$fh - gk = fh - gh + gh - gk = (f - g)h + (h - k)g \in I.$$

Hence, $(f + h, g + k), (fh, gk) \in \rho_I$. Therefore, ρ_I is a congruence relation on $\mathcal{R}(L_\tau)$. Also, from

$$h \in I_{\rho_I} \Leftrightarrow \text{ there exists } (f, g) \in \rho_I(h = f - g) \Rightarrow h \in I$$

and

$$h \in I \Rightarrow (2h, h) \in \rho_I \Rightarrow h \in I_{\rho_I},$$

we infer that $I = I_{\rho_I}$.

Let $f, g \in \mathcal{R}(L_\tau)$ with $z(f - g) \in E(\rho_I)$ be given. Then there exists an element $(h, k) \in \rho_I$ such that $z(f - g) = z(h - k)$. Since I is a z-ideal of $\mathcal{R}(L_\tau)$ and $h - k \in I$, we infer that $f - g \in I$, which implies that $(f, g) \in \rho_I$. Hence, ρ_I is a z-congruence on $\mathcal{R}(L_\tau)$. □

Proposition 12 *Let L_τ be a topoframe. Then, the following statements are true:*

(1) *If ρ is a proper congruence on $\mathcal{R}^+(L_\tau)$, then I^+_ρ is a proper ideal of $\mathcal{R}^+(L_\tau)$ and $\rho = \rho^+_{I^+_\rho}$.*

(2) *If I is a proper ideal of $\mathcal{R}^+(L_\tau)$, then ρ^+_I is a proper congruence on $\mathcal{R}^+(L_\tau)$ and $I = I^+_{\rho^+_I}$. In particular, if I is a z-ideal of $\mathcal{R}^+(L_\tau)$, then ρ^+_I is a z-congruence on $\mathcal{R}^+(L_\tau)$.*

Proof:

(1) Let $v, w \in I^+_\rho$ be given. Then there exist $(f, g), (h, k) \in \rho$ such that $v = f - g$ and $w = h - k$, which implies that $(f + k, g + h) \in \rho$ and

$v-w=(f+k)-(g+h)$; that is, $v-w\in I^+_\rho$. Now, assume that $(v,w)\in I^+_\rho\times\mathcal{R}^+(L_\tau)$. Thus, there exists an element (f,g) in ρ such that $v=f-g$, which implies that $vw=fw-gw$ and $(fw,gw)\in\rho$, and so, $vw\in I^+_\rho$. Therefore, I^+_ρ is an ideal of $\mathcal{R}^+(L_\tau)$. Also, from

$$(f,g)\in\rho^+_{I^+_\rho}\Leftrightarrow f-g\in I^+_\rho\Leftrightarrow (f,g)\in\rho,$$

we infer that $\rho=\rho^+_{I^+_\rho}$.

(2) It is evident that ρ^+_I is an equivalence relation on $\mathcal{R}^+(L_\tau)$. Let $(f,g),(h,k)\in\rho^+_I$ be given. Then $f-g,h-k\in I$, which implies that $(f+h)-(g+k)\in I$ and

$$fh-gk=fh-gh+gh-gk=(f-g)h+(h-k)g\in I.$$

Hence, $(f+h,g+k),(fh,gk)\in\rho^+_I$. Therefore, ρ^+_I is a congruence relation on $\mathcal{R}^+(L_\tau)$. Also, from

$$h\in I^+_{\rho^+_I}\Leftrightarrow \text{ there exists } (f,g)\in\rho^+_I(h=f-g)\Rightarrow h\in I$$

and

$$h\in I\Rightarrow(2h,h)\in\rho^+_I\Rightarrow h\in I^+_{\rho^+_I},$$

we infer that $I=I^+_{\rho^+_I}$.

Let $f,g\in\mathcal{R}^+(L_\tau)$ with $z(f-g)\in E(\rho^+_I)$ be given. Then there exists an element $(h,k)\in\rho^+_I$ such that $z(f-g)=z(h-k)$. Since I is a z-ideal of $\mathcal{R}^+(L_\tau)$ and $h-k\in I$, we infer that $f-g\in I$, which implies that $(f,g)\in\rho^+_I$. Hence, ρ^+_I is a z-congruence on $\mathcal{R}^+(L_\tau)$. □

Let ρ be a congruence relation on $\mathcal{R}(L_\tau)$. We set

(1) $\mathrm{MaxCong}(\mathcal{R}(L_\tau)):=\{\rho\colon \rho \text{ is a maximal congruence on } \mathcal{R}(L_\tau)\}$ and $\mathrm{MaxId}(\mathcal{R}(L_\tau)):=\{M\colon M \text{ is a maximal ideal of } \mathcal{R}(L_\tau)\}$.

(2) $\mathfrak{MCong}(f,g):=\{\rho\in\mathrm{MaxCong}(\mathcal{R}(L_\tau))\colon (f,g)\in\rho\}$ for every $f,g\in\mathcal{R}(L_\tau)$.

(3) $\mathfrak{MCong}^+(f,g):=\{\rho\in\mathrm{MaxCong}(\mathcal{R}^+(L_\tau))\colon (f,g)\in\rho\}$ for every $f,g\in\mathcal{R}^+(L_\tau)$.

Let S be a subset of ring $\mathcal{R}(L_\tau)$. We define the hull of S by

$$h_{\text{Max}}(S) := \{M \in \text{MaxId}(\mathcal{R}(L_\tau)) \colon S \subseteq M\}.$$

For any subset T of $\text{MaxId}(\mathcal{R}(L_\tau))$, the kernel of T is defined as

$$k_{\text{Max}}(T) := \bigcap T.$$

Also for every f of $\mathcal{R}(L_\tau)$, we put $h_{\text{Max}}(f) := h_{\text{Max}}(\{f\})$ and $k_{\text{Max}}(f) := k_{\text{Max}}(\{f\})$.

The following proposition shows that there is a one-one correspondence between maximal congruence relation on a ring $\mathcal{R}(L_\tau)$ and a maximal ideal of the ring $\mathcal{R}(L_\tau)$.

Proposition 13 *Let L_τ be a topoframe. Then, the following statements are true:*

(1) *If ρ is a maximal congruence on $\mathcal{R}(L_\tau)$, then ρ is a z-congruence on $\mathcal{R}(L_\tau)$ and I_ρ is a maximal ideal of $\mathcal{R}(L_\tau)$.*

(2) *If M is a maximal ideal of $\mathcal{R}(L_\tau)$, then ρ_M is a maximal congruence on $\mathcal{R}(L_\tau)$.*

(3) *The map $\theta(\rho \mapsto z(I_\rho)) \colon \text{zCong}(L_\tau) \to \text{zFil}(L_\tau)$ is a bijection function.*

(4) *The map $\theta(\rho \mapsto I_\rho) \colon \text{MaxCong}(\mathcal{R}(L_\tau)) \to \text{MaxId}(\mathcal{R}(L_\tau))$ is a bijection function and $\theta(\mathfrak{MCong}(f,g)) = h_{\text{Max}}(f-g)$ for every $f, g \in \mathcal{R}(L_\tau)$, where $h_{\text{Max}}(f-g) := \{M \in \text{MaxId}(\mathcal{R}(L_\tau)) \colon f-g \in M\}$.*

Proof:

(1) Let Q be a proper ideal of $\mathcal{R}(L_\tau)$ such that $I_\rho \subseteq Q$. Then

$$\begin{aligned} \rho = \rho_{I_\rho} \subseteq \rho_Q \Rightarrow \rho = \rho_Q, &\quad \text{since } \rho \text{ is a maximal congruence on } \mathcal{R}(L_\tau) \\ \Rightarrow I_\rho = I_{\rho_Q} = Q. & \end{aligned}$$

Hence, I_ρ is a maximal ideal of $\mathcal{R}(L_\tau)$. Also, by Proposition 11, $\rho = \rho_{I_\rho}$ is a z-congruence on $\mathcal{R}(L_\tau)$ since I_ρ is a z-ideal of $\mathcal{R}(L_\tau)$.

(2) Let ρ be a congruence relation on $\mathcal{R}(L_\tau)$ such that $\rho_M \subseteq \rho$. Then by Proposition 11

$$\begin{aligned} M = M_{\rho_M} \subseteq M_\rho \Rightarrow M = M_\rho, &\quad \text{since } M \text{ is a maximal ideal of } \mathcal{R}(L_\tau) \\ \Rightarrow \rho_M = \rho_{M_\rho} = \rho. & \end{aligned}$$

Hence, ρ_M is a maximal congruence on $\mathcal{R}(L_\tau)$.

(3) It is evident by part (1) and Proposition 10.

(4) Let ρ be a maximal congruence on $\mathcal{R}(L_\tau)$ such that $(f,g)\in\rho$. Then by (1), I_ρ is a maximal ideal of $\mathcal{R}(L_\tau)$ and $f-g\in I_\rho$. Therefore, $I_\rho \in h_{\text{Max}}(f-g)$. Then $\theta\big(\mathfrak{MCong}(f,g)\big)\subseteq h_{\text{Max}}(f-g)$. Now, let M be a maximal ideal and $f-g\in M$. By part (2), ρ_M is a maximal congruence and $(f,g)\in\rho_M$. Then $M=M_{\rho_M}=\theta(\rho)\in\theta\big(\mathfrak{MCong}(f,g)\big)$. □

Proposition 14 *Let L_τ be a topoframe. The following statements are true:*

(1) *If ρ is a maximal congruence on $\mathcal{R}^+(L_\tau)$, then I^+_ρ is a maximal ideal of $\mathcal{R}^+(L_\tau)$.*

(2) *If M is a maximal ideal of $\mathcal{R}^+(L_\tau)$, then ρ^+_M is a maximal congruence on $\mathcal{R}^+(L_\tau)$.*

(3) *The map $\theta\big(\rho\mapsto z[I^+_\rho]\big)\colon \mathrm{zCong}^+(L_\tau)\to\mathrm{zFil}(L_\tau)$ is a bijection function.*

(4) *The map $\theta\big(\rho\mapsto I^+_\rho\big)\colon \mathrm{zCong}^+\big(\mathcal{R}^+(L_\tau)\big)\to \mathrm{MaxId}^+\big(\mathcal{R}^+(L_\tau)\big)$ is a bijection function and $\theta\big(\mathfrak{MCong}^+(f,g)\big)=h^+_{\text{Max}}(f-g)$ for every $f,g\in\mathcal{R}^+(L_\tau)$, where $h^+_{\text{Max}}(f-g):=\big\{M\in\mathrm{MaxId}\big(\mathcal{R}^+(L_\tau)\big)\colon f-g\in M\big\}$.*

Proof:

(1) Let Q be a proper ideal of $\mathcal{R}^+(L_\tau)$ such that $I^+_\rho\subseteq Q$. Then

$$\begin{aligned}\rho=\rho^+_{I^+_\rho}\subseteq\rho^+_Q\Rightarrow\rho=\rho^+_Q,\quad&\text{since }\rho\text{ is a maximal congruence on }\mathcal{R}^+(L_\tau)\\ \Rightarrow I^+_\rho=I^+_{\rho^+_Q}=Q.&\end{aligned}$$

Hence, I^+_ρ is a maximal ideal of $\mathcal{R}^+(L_\tau)$.

(2) Let ρ be a congruence relation on $\mathcal{R}^+(L_\tau)$ such that $\rho^+_M\subseteq\rho$. Then

$$\begin{aligned}M=M^+_{\rho^+_M}\subseteq M^+_\rho\Rightarrow M=M^+_\rho,\quad&\text{since }M\text{ is a maximal ideal of }\mathcal{R}^+(L_\tau)\\ \Rightarrow \rho^+_M=\rho^+_{M^+_\rho}=\rho.&\end{aligned}$$

Hence, ρ is a maximal congruence on $\mathcal{R}^+(L_\tau)$.

(3) Let ρ be a z-congruence relation on $\mathcal{R}^+(L_\tau)$. Then by Proposition 10, $z(I^+_\rho)$ is a z-filter on L_τ. Now, let $\mathcal{F}$ be a z-filter on L_τ. It is easy to see that $E^{-1}(\mathcal{F})$ is a z-congruence and

$$\begin{aligned}\theta(E^{-1}(\mathcal{F})) &= E^+(E^{-1}(\mathcal{F}))\\ &= \{z(f-g)\colon f,g \in \mathcal{R}^+(L_\tau) \text{ and } (f,g) \in E^{-1}(\mathcal{F})\}\\ &= \{z(f-g)\colon f,g \in \mathcal{R}^+(L_\tau) \text{ and } z(f-g) \in \mathcal{F}\}\\ &= \mathcal{F}.\end{aligned}$$

Then θ is an onto function.

(4) By (3), for every maximal congruence ρ, I^+_ρ is a maximal ideal of $\mathcal{R}^+(L_\tau)$. Now, let I be a maximal ideal of $\mathcal{R}^+(L_\tau)$. Then by (4), ρ^+_I is a maximal congruence and

$$\theta(\rho^+_I) = I^+_{\rho^+_I} = I.$$

Therefore θ is a bijection function. □

Now, we give the relationships between z-congruences on a ring $\mathcal{R}(L_\tau)$ and z-congruence relations on a semiring $\mathcal{R}^+(L_\tau)$.

Proposition 15 *Let L_τ be a topoframe. Then, the following properties hold:*

(1) *If ρ is a congruence (or a z-congruence) relation on $\mathcal{R}(L_\tau)$, then $\rho^c := \rho \cap \big(\mathcal{R}^+(L_\tau) \times \mathcal{R}^+(L_\tau)\big)$ is a congruence (or a z-congruence) relation on $\mathcal{R}^+(L_\tau)$.*

(2) *If ρ is a z-congruence relation on $\mathcal{R}(L_\tau)$, then $E(\rho) = E^+(\rho^c)$.*

(3) *If ρ is a congruence (or a z-congruence) relation on $\mathcal{R}^+(L_\tau)$, then $\rho^e := \{(f,g)\colon f,g \in \mathcal{R}(L_\tau) \text{ and } f-g = h-k \text{ for some } (h,k) \in \rho\}$ is a congruence (or a z-congruence) relation on $\mathcal{R}(L_\tau)$ and*

$$E(\rho) = E(\rho^{ec}) = E(\rho^e).$$

Proof:

(1) It is evident that ρ^c is a congruence relation on $\mathcal{R}^+(L_\tau)$. Let $f,g \in \mathcal{R}^+(L_\tau)$ with $z(f-g) \in E^+(\rho^c)$ be given. Then $z(f-g) \in E(\rho)$, which implies that

$$(f,g) \in \rho \cap \mathcal{R}^+(L_\tau) \times \mathcal{R}^+(L_\tau) = \rho^c.$$

(2) It is evident that $E^+(\rho^c) \subseteq E(\rho)$. Let $f, g \in \mathcal{R}(L_\tau)$ with $(f, g) \in \rho$ be given. Since ρ is the z-congruence relation on $\mathcal{R}(L_\tau)$ and $z(|f-g|-\mathbf{0}) = z(f-g) \in E(\rho)$, we conclude that $(|f-g|, \mathbf{0}) \in \rho \cap \mathcal{R}^+(L_\tau) \times \mathcal{R}^+(L_\tau) = \rho^c$. Hence, $E(\rho) \subseteq E^+(\rho^c)$.

(3) Let $f, g, h \in \mathcal{R}(L_\tau)$ with $(f, g), (g, h) \in \rho^e$ be given. Then there exist $r, s, v, w \in \mathcal{R}^+(L_\tau)$ such that $f - g = r - s$, $g - h = v - w$, and $(r, s), (v, w) \in \rho$, which implies that $f - h = (r + v) - (s + w)$ and $(r+v, s+w) \in \rho$. Thus $(f, h) \in \rho^e$. Then ρ^e is an equivalence relation on $\mathcal{R}(L_\tau)$. It is clear that if $(f, g), (h, k) \in \rho^e$, then $(f+h, g+k), (fh, gk) \in \rho^e$, which implies that ρ^e is a congruence relation on $\mathcal{R}(L_\tau)$.

Since $\rho \subseteq \rho^{ec} \subseteq \rho^e$, we infer that $E(\rho) \subseteq E(\rho^{ec}) \subseteq E(\rho^e)$. Let $f, g \in \mathcal{R}(L_\tau)$ with $(f, g) \in \rho^e$ be given. Then there exists an element (h, k) in ρ such that $f-g = h-k$, which implies that $z(f-g) = z(h-k)$. Hence, $E(\rho^e) \subseteq E(\rho)$; that is, $E(\rho) = E(\rho^{ec}) = E(\rho^e)$.

Suppose that ρ is a z-congruence relation on $\mathcal{R}^+(L_\tau)$. Let $f, g \in \mathcal{R}(L_\tau)$ with $z(f - g) \in E(\rho^e)$ be given. It is evident that the map $\theta : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ given by

$$\theta(\{r\}) = \begin{cases} \{2r\} & \text{if } r > 0 \\ (-\infty, 0] & \text{if } r = 0, \\ \emptyset & \text{if } r < 0. \end{cases}$$

is a frame map. We set $h := (f - g) \circ \theta \in \mathcal{R}(L_\tau)$ and $k := (g - f) \circ \theta \in \mathcal{R}(L_\tau)$. From

$$\operatorname{coz}(h) = \operatorname{coz}\big((f-g) \circ \theta\big) = (f-g)\big(\operatorname{coz}(\theta)\big) = (f-g)\big(\theta(\mathbb{R}^+)\big) = h(\mathbb{R}^+),$$

we conclude that $h \in \mathcal{R}^+(L_\tau)$. A similar argument shows that $k \in \mathcal{R}^+(L_\tau)$. From

$$\begin{aligned} (h - k)(\{r\}) &= \bigvee \big\{h(\{a\}) \wedge (-k)(\{b\}) \colon a + b = r\big\} \\ &= \bigvee \big\{h(\{a\}) \wedge k(\{-b\}) \colon a + b = r\big\} \\ &= \bigvee \big\{(f - g)\big(\theta(\{a\})\big) \wedge (g - f)\big(\theta(\{a - r\})\big) \colon a \in \mathbb{R}\big\} \\ &= \bigvee \big\{(f - g)\big(\theta(\{a\})\big) \wedge (f - g)\big(\theta(\{r - a\})\big) \colon a \in \mathbb{R}\big\} \\ &= \bigvee \big\{(f - g)\big(\theta(\{a\} \cap \{r - a\})\big) \colon a \in \mathbb{R}\big\} \\ &= (f - g)\big(\theta(\{r/2\})\big) \\ &= (f - g)(\{r\}) \end{aligned}$$

for every $r \in \mathbb{R}$, we infer that $f - g = h - k$, which implies that

$$z(h-k) = z(f-g) \in E(\rho^e) = E(\rho).$$

Since ρ is the z-congruence relation on $\mathcal{R}^+(L_\tau)$, then $(h, k) \in \rho$, which implies that $(f, g) \in \rho^e$. Therefore, ρ^e is a z-congruence on $\mathcal{R}(L_\tau)$.□

In the following propositions, we examine the equivalent conditions for a congruence relation on $\mathcal{R}(L_\tau)$ or a semiring $\mathcal{R}^+(L_\tau)$ to become a z-congruence relation.

Proposition 16 *The following statements are equivalent for a congruence relation ρ on $\mathcal{R}(L_\tau)$:*

(1) *ρ is a z-congruence relation on $\mathcal{R}(L_\tau)$.*

(2) $\operatorname{coz}^{-1}\operatorname{coz}(I_\rho) = I_\rho$.

(3) *For every $f \in I_\rho$, $\operatorname{coz}^{-1}\big(\operatorname{coz}(\{f\})\big) \subseteq I_\rho$.*

(4) *For every $f \in I_\rho$, $\operatorname{coz}^{-1}\big(\downarrow \operatorname{coz}(f)\big) \subseteq I_\rho$.*

(5) $I_\rho = \bigcup_{f \in I_\rho} M_{\operatorname{coz}(f)}$.

(6) $\rho = \big\{(f,g) \colon f, g \in \mathcal{R}(L_\tau)$ *and* $\operatorname{coz}(f-g) \leq \operatorname{coz}(h)$ *for some* $h \in I_\rho\big\}$.

Proof: By Proposition 11 and [8, Proposition 3.1], the proof is straightforward. □

Proposition 17 *The following statements are equivalent for a congruence relation ρ of $\mathcal{R}^+(L_\tau)$:*

(1) *ρ is a z-congruence relation on $\mathcal{R}^+(L_\tau)$.*

(2) $\operatorname{coz}^{-1}\operatorname{coz}(I_\rho^+) = I_\rho^+$.

(3) *For every $f \in I_\rho^+$, $\operatorname{coz}^{-1}\big(\operatorname{coz}(\{f\})\big) \subseteq I_\rho^+$.*

(4) *For every $f \in I_\rho^+$, $\operatorname{coz}^{-1}\big(\downarrow \operatorname{coz}(f)\big) \subseteq I_\rho^+$.*

(5) $I_\rho^+ = \bigcup_{f \in I_\rho^+} M_{\operatorname{coz}(f)}$.

(6) $\rho = \big\{(f,g) \colon f, g \in \mathcal{R}^+(L_\tau)$ *and* $\operatorname{coz}(f-g) \leq \operatorname{coz}(h)$ *for some* $h \in I_\rho^+\big\}$.

Proof:

(1) ⇒ (2) Let ρ be a z-congruence relation on $\mathcal{R}^+(L_\tau)$ and $f \in \operatorname{coz}^{-1}\operatorname{coz}(I^+_\rho)$. Then $coz(f) \in \operatorname{coz}(I^+_\rho)$ and so there exists $(g,h) \in \rho$ such that $coz(f) = coz(g-h)$. Therefore, $z(f-0) = z(f) = z(g-h) \in E(\rho)$ and so $(f,0) \in \rho$ since ρ is a z-congruence relation on $\mathcal{R}^+(L_\tau)$. Hence $f = f - 0 \in I^+_\rho$.

(2) ⇒ (3) Assume $f \in I^+_\rho$ and $g \in \operatorname{coz}^{-1}\big(\operatorname{coz}(\{f\})\big)$. Since $\operatorname{coz}^{-1}\big(\operatorname{coz}(\{f\})\big) \subseteq \operatorname{coz}^{-1}\operatorname{coz}(I^+_\rho)$, we have $g \in \operatorname{coz}^{-1}\operatorname{coz}(I^+_\rho)$. Therefore, by (2), $g \in I^+_\rho$.

(3) ⇒ (4) Assume $g \in \operatorname{coz}^{-1}\big(\downarrow\operatorname{coz}(f)\big)$. Then $\operatorname{coz}(g) \leq \operatorname{coz}(f)$, and so $\operatorname{coz}(g) = \operatorname{coz}(f) \wedge \operatorname{coz}(g) = \operatorname{coz}(fg)$. By Proposition 12, I^+_ρ is an ideal and $f \in I^+_\rho$. Hence we conclude that $fg \in I^+_\rho$. Therefore, by (3), $g \in I^+_\rho$.

(4) ⇒ (5) Clearly, $I^+_\rho \subseteq \bigcup_{f \in I^+_\rho} M_{\operatorname{coz}(f)}$ because, for any $g \in I^+_\rho$, $g \in M_{\operatorname{coz}(g)}$. To see the reverse inclusion, let $f \in I^+_\rho$, and consider any $g \in M_{\operatorname{coz}(f)}$. This means $\operatorname{coz}(g) \leq \operatorname{coz}(f)$. Therefore, by (4), $g \in I^+_\rho$ shows that $M_{\operatorname{coz}(f)} \subseteq I^+_\rho$, and hence the desired inclusion.

(5) ⇒ (6) Let $(f,g) \in \rho$. Then $f - g \in I^+_\rho$. By (5), there exists $h \in I^+_\rho$ such that $f - g \in M_{\operatorname{coz}(h)}$, and so $\operatorname{coz}(f-g) \leq \operatorname{coz}(h)$.

(6) ⇒ (1) It is evident by definition of ρ. □

4 k-congruences on a Ring $\mathcal{R}(L_\tau)$ and a Semiring $\mathcal{R}^+(L_\tau)$

A ring A is said to be regular if for every $x \in A$ there is $y \in A$ with $x = x^2y$. We recall from [4, Definition 3.1] that if $z[L_\tau] \subseteq \tau$, then L_τ is called a P-topoframe. In the following proposition, we investigate, the equivalent conditions for a topoframe to become a P-topoframe.

Proposition 18 *Let L_τ be a topoframe. The following statements are equivalent:*

(1) *L_τ is a P-topoframe.*

(2) *$\mathcal{R}(L_\tau)$ is a regular ring.*

(3) *Every ideal of $\mathcal{R}(L_\tau)$ is a z-ideal à la Mason.*

(4) *Every ideal of* $\mathcal{R}(L_\tau)$ *is a* z*-ideal of* $\mathcal{R}(L_\tau)$.

(5) *Every congruence on* $\mathcal{R}(L_\tau)$ *is a* z*-congruence relation on* $\mathcal{R}(L_\tau)$.

(6) *Every cancellative congruence on* $\mathcal{R}^+(L_\tau)$ *is a* z*-congruence relation on* $\mathcal{R}^+(L_\tau)$.

(7) *Every* k*-congruence on* $\mathcal{R}^+(L_\tau)$ *is a* z*-congruence relation on* $\mathcal{R}^+(L_\tau)$.

(8) *Every* k*-ideal in* $\mathcal{R}^+(L_\tau)$ *is a semiprime of* $\mathcal{R}^+(L_\tau)$.

(9) $\mathcal{R}^+(L_\tau)$ *is a regular semiring.*

Proof:

(1) $\Leftrightarrow$ (2) By [4, Theorem 3.5], L_τ is a P-topoframe if and only if $\mathcal{R}(L_\tau)$ is a regular ring.

(2) $\Leftrightarrow$ (3) By [12, Theorem 1.2], $\mathcal{R}(L_\tau)$ is a regular ring if and only if every ideal of $\mathcal{R}(L_\tau)$ is a z-ideal à la Mason.

(3) $\Rightarrow$ (4) By [7, Proposition 6.2], every z-ideal *à* la Mason of $\mathcal{R}(L_\tau)$ is a z-ideal of $\mathcal{R}(L_\tau)$.

(4) $\Rightarrow$ (5) Let ρ be a congruence on $\mathcal{R}(L_\tau)$. Then, by Proposition 11, I_ρ is an ideal of $\mathcal{R}(L_\tau)$, which implies from our hypothesis that I_ρ is a z-ideal of $\mathcal{R}(L_\tau)$. Thus, by Proposition 11, ρ_{I_ρ} is a z-congruence on $\mathcal{R}(L_\tau)$. From

$$(f,g) \in \rho_{I_\rho} \Leftrightarrow f-g \in I_\rho \Leftrightarrow (f,g) \in \rho,$$

we conclude that ρ is a z-congruence on $\mathcal{R}(L_\tau)$.

(5) $\Rightarrow$ (6) Let ρ be a cancellative congruence on $\mathcal{R}^+(L_\tau)$. Then, by part (3) of Proposition 15, ρ^e is a congruence relation on $\mathcal{R}(L_\tau)$, which implies from our hypothesis that ρ^e is a z-congruence relation on $\mathcal{R}(L_\tau)$. It is evident that $\rho \subseteq \rho^{ec}$. Let $f, g \in \mathcal{R}^+(L_\tau)$ with $(f,g) \in \rho^{ec}$. From

$$\begin{aligned}(f,g) \in \rho^e &\Rightarrow \exists\, h,k \in \mathcal{R}^+(L_\tau)\big((h,k) \in \rho \text{ and } f-g = h-k\big)\\ &\Rightarrow \exists\, h,k \in \mathcal{R}^+(L_\tau)\big((f-g+k,k) = (h,k) \in \rho \text{ and}\\ &\quad f-g = h-k\big)\\ &\Rightarrow \exists\, h,k \in \mathcal{R}^+(L_\tau)\big((f-g,\mathbf{0}) \in \rho \,\text{and}\, f-g = h-k\big),\\ &\quad \text{by our hypothesis}\\ &\Rightarrow (f,g) \in \rho,\end{aligned}$$

we conclude that $\rho = \rho^{ec}$. Let $f, g \in \mathcal{R}^+(L_\tau)$ with $z(f-g) \in E^+(\rho)$ be given. Thus

$$\begin{aligned} z(f-g) \in E(\rho^e) &\Rightarrow (f,g) \in \rho^e, \qquad \text{since } \rho^e \text{ is } z\text{-congruence} \\ &\Rightarrow (f,g) \in \rho^{ec} = \rho, \end{aligned}$$

and so ρ is a z-congruence relation on $\mathcal{R}^+(L_\tau)$.

(6) $\Rightarrow$ (7) Let ρ be a k-congruence relation on $\mathcal{R}^+(L_\tau)$. Thus, there is an ideal I of $\mathcal{R}^+(L_\tau)$ such that $\rho = k_I^+$. Let $f, g, h \in \mathcal{R}^+(L_\tau)$ with $(f+h, g+h) \in \rho$ be given. Thus, there exist $v, w \in I$ such that $f+h+v = g+h+w$, which implies that $f+v = g+w$, and so, $(f,g) \in k_I^+ = \rho$. Then, ρ is a cancellative congruence on $\mathcal{R}^+(L_\tau)$, and by our hypothesis, we obtain that ρ is a z-congruence relation on $\mathcal{R}^+(L_\tau)$.

(7) $\Rightarrow$ (8) Let I be a k-ideal in $\mathcal{R}^+(L_\tau)$. It is evident that ρ_I^+ is an equivalence relation on $\mathcal{R}^+(L_\tau)$. Let $f, g, h, k \in \mathcal{R}^+(L_\tau)$ with $(f,g), (h,k) \in \rho_I^+$ be given. Then $f-g, h-k \in I$, which implies that $f-g+h-k \in I$ and $fh - gk = (f-g)h + (h-k)g \in I$. Hence, $(f+h, g+k), (fh, gk) \in \rho_I^+$. Therefore, ρ_I^+ is a congruence relation on $\mathcal{R}^+(L_\tau)$. For every $f, g \in \mathcal{R}^+(L_\tau)$, we have

$$\begin{aligned} (f,g) \in \rho_I^+ &\Rightarrow \exists\, h \in I(f-g=h) \Rightarrow \exists\, h \in I(f+\mathbf{0} = g+h) \\ &\Rightarrow (f,g) \in k_I^+ \end{aligned}$$

and

$$\begin{aligned} (f,g) \in k_I^+ &\Rightarrow \text{ there exist } h, k \in I(f+h = g+k) \\ &\Rightarrow \text{ there exist } h, k \in I(f-g+h = k) \\ &\Rightarrow f-g \in I, \qquad \text{since } I \text{ is a } k\text{-ideal} \\ &\Rightarrow (f,g) \in \rho_I^+. \end{aligned}$$

Hence, $\rho_I^+ = k_I^+$; that is, ρ_I^+ is a k-congruence relation on $\mathcal{R}^+(L_\tau)$ and by our hypothesis, ρ_I^+ is a z-congruence relation on $\mathcal{R}^+(L_\tau)$. If $(n, f) \in \mathbb{N} \times \mathcal{R}^+(L_\tau)$ with $f^n \in I$, then from $z(f-\mathbf{0}) = z(f^n - \mathbf{0}) \in E(\rho_I^+)$, we infer that $(f, \mathbf{0}) \in \rho_I^+$ that is, $f = f - \mathbf{0} \in I$. Therefore, I is a semiprime of $\mathcal{R}^+(L_\tau)$.

(8) $\Rightarrow$ (9) By [14, Theorem 3.2], $\mathcal{R}^+(L_\tau)$ is a regular semiring.

(9) ⇒ (1) Let $f \in \mathcal{R}(L_\tau)$ be given. Then, by our hypothesis, there exists an element g in $\mathcal{R}^+(L_\tau)$ such that $|f| = f^2 g$, which implies that $(|f|g)^2 = |f|g$ is an idempotent element of $\mathcal{R}(L_\tau)$. Thus, by [4, Remark 3.3], $z(f) = z(f^2 g) = z(|f|g) \in \tau$. Hence, $z[L_\tau] \subseteq \tau$. □

Proposition 19 *Let L_τ be a topoframe. Then the following statements are equivalent:*

(1) *For every $f, g, h, k \in \mathcal{R}(L_\tau)$, if $\mathfrak{MCong}(f, g) \subseteq \mathfrak{MCong}(h, k)$, then $z(f - g) \leq z(h - k)$.*

(2) *For every $f, g \in \mathcal{R}(L_\tau)$, if $\mathfrak{MCong}(f, \mathbf{0}) \subseteq \mathfrak{MCong}(g, \mathbf{0})$, then $z(f) \leq z(g)$.*

(3) *For every $f, g \in \mathcal{R}(L_\tau)$, if $h_{\text{Max}}(f) \subseteq h_{\text{Max}}(g)$, then $\operatorname{coz}(g) \leq \operatorname{coz}(f)$.*

(4) *For every $f \in \mathcal{R}(L_\tau)$, $\bigcap h_{\text{Max}}(f) \subseteq M_{\operatorname{coz}(f)}$.*

(5) *For every $f \in \mathcal{R}(L_\tau)$, $\bigcap h_{\text{Max}}(f) = M_{\operatorname{coz}(f)}$.*

(6) *Every z-ideal of $\mathcal{R}(L_\tau)$ is a z-ideal à la Mason of $\mathcal{R}(L_\tau)$.*

(7) *If ρ is a z-congruence on $\mathcal{R}(L_\tau)$, then for every $(f, g) \in \rho$,*

$$\bigcap \mathfrak{MCong}(f, g) \subseteq \rho.$$

Proof:

(1) ⇔ (2) Since for every congruence ρ on $\mathcal{R}(L_\tau)$, $(f, g) \in \rho$ if and only if $(f - g, \mathbf{0}) \in \rho$, so the proof is evident.

(2) ⇔ (3) By Proposition 11, the proof is straightforward.

(3) ⇔ (4) ⇔ (5) ⇔ (6) See [8, Proposition 3.2].

(1) ⇒ (7) Let ρ be a z-congruence on $\mathcal{R}(L_\tau)$ and $(f, g) \in \rho$. If $(h, k) \in \bigcap \mathfrak{MCong}(f, g)$, then $\mathfrak{MCong}(f, g) \subseteq \mathfrak{MCong}(h, k)$, which implies from our hypothesis that $z(f - g) \leq z(h - k)$. Since $z(fh + gk - fk - gh) = z\big((f - g)(h - k)\big) = z(h - k)$ and $(fh + gk, fk + gh) \in \rho$, we infer that $(h, k) \in \rho$ as ρ is a z-congruence.

(7) ⇒ (6) Suppose that I is a z-ideal of $\mathcal{R}(L_\tau)$. Let $(f,g) \in I \times \mathcal{R}(L_\tau)$ with $h_{\text{Max}}(f) = h_{\text{Max}}(g)$ be given. Since $(f,\mathbf{0}) \in \rho_I$, and by Proposition 11, $\mathfrak{MCong}(f,\mathbf{0}) = \mathfrak{MCong}(g,\mathbf{0})$, we infer from our hypothesis that

$$(g,\mathbf{0}) \in \bigcap \mathfrak{MCong}(g,\mathbf{0}) \subseteq \rho_I,$$

which implies that $g \in I$. Hence, I is a z-ideal *à* la Mason of $\mathcal{R}(L_\tau)$.

□

Proposition 20 *Let L_τ be a topoframe. Then, the following statements are true:*

(1) *If ρ is a proper congruence on $\mathcal{R}(L_\tau)$ and $\bigcap \mathfrak{MCong}(f,g) \subseteq \rho$ for every $(f,g) \in \rho$, then ρ is a z-congruence on $\mathcal{R}(L_\tau)$.*

(2) *If ρ is a proper congruence on $\mathcal{R}^+(L_\tau)$ and $\bigcap \mathfrak{MCong}^+(f,g) \subseteq \rho$ for every $(f,g) \in \rho$, then ρ is a z-congruence on $\mathcal{R}^+(L_\tau)$.*

Proof:

(1) Let $f,g,h,k \in \mathcal{R}(L_\tau)$ with $(f,g) \in \rho$ and $z(f-g) = z(h-k)$ be given. If μ is a maximal congruence on $\mathcal{R}(L_\tau)$ and $(f,g) \in \mu$, then $(h,k) \in \mu$ since by Proposition 13, every maximal congruence is a z-congruence. Thus,

$$(h,k) \in \bigcap \mathfrak{MCong}(f,g) \subseteq \rho,$$

and we obtain that ρ is a z-congruence on $\mathcal{R}(L_\tau)$.

(2) The proof is similar to the proof of part (1).

□

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