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Higher Order mz-elements in Coherent Quantales

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Abstract

The mz-elements of a coherent quantale have recently been defined by the author as an abstraction of the mz-ideals of a unital commutative ring.

Having as its starting point the Dube and Ighedo recent paper on higher order ideals in ring theory, this paper deals with the higher order mz-elements of a coherent quantale A. For each natural number n we define the mz^n-elements of A, so we obtain an ascending sequence that covers the set of all higher order mz-elements. We obtain a lot of properties of this sequence. In particular, the stationarity of the sequence is studied. Another category of results investigates how the coherent quantale morphisms preserve such properties.

Keywords: coherent quantales, mz-elements, mz^n-elements, mz-terminating quantale, radically mz-covered quantale

1 Introduction

The notion of z-ideal has been introduced by Kohls in [16], but it was Mason who defined and studied the z-ideals of a (unital) commutative ring. Mason's article was followed by numerous articles on z-ideals and some extensions thereof. These include Dube and Ighedo's [9] work on z^n-ideals of a commutative ring R (n is an arbitrary positive integer). The article of Dube and Ighedo is continued by Benhissi and Maatallah with new results on z^n-ideals in commutative rings [4, 17].

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Quantales were introduced by Mulvey in [22] as an abstract framework for studying the fundamentals of quantum mechanics and non-commutative $C^{\star}$-algebras (see also [10, 26]). An important example of quantale is the multiplicative lattice $Id(R)$ of all ideals of a unital ring R. An impressive number of papers have generalized the algebraic and topological properties of $Id(R)$ to properties of any or certain type of quantale. In this way, quantale theory can be viewed as part of Abstract Ideal Theory (in the sense of [5]). At the same time, the quantales generalize the frames, structures in which the point-free topology is developed (see [15, 25]).

Inspired by Martinez and Zenk's work [20], Ighedo and McGovern define in [14] the notion of mz-ideals of a commutative ring. The relationship between z-ideals and mz-ideals is not clear: each mz-ideal is a z-ideal, but it is unknown if the converse assertion holds (see the discussion in Example 4.2 of [14]). In the mentioned paper, a lot of mz-ideals properties are obtained by using the frame theory.

In [13] we defined the mz-elements of a coherent quantale A as an abstraction of the mz-ideals of a commutative ring.

Among the open problems listed in the conclusions section of the paper [8] there is also the study of the higher order z-elements of a multiplicative lattice. Then the definition and study of a notion of higher order mz-element of a quantale can be considered as a neighboring open problem. This paper is an answer to this problem.

Taking as a model the notion of z^n-ideal from [14], we will define the mz^n-elements of a quantale A. Although the definition of an mz-element is almost a translation of the definition of an z^n-ideal, when we apply it to the particular case of the quantale $Id(R)$ we don't quite get the concept of z^n-ideal. An mz^n-element of the quantale $Id(R)$ will be called an mz^n-ideal. Any mz^n-ideal is a z^n-ideal, but we don't know if the converse statement is true.

Now we shall describe the content of the paper. Section 2 contains some background facts on quantales: m-prime and maximal elements of an algebraic quantale A, the radical $\rho(a)$ of an element $a \in A$ and its basic properties, etc. We recall from [19] a fundamental result that describes the form of $\rho(a)$ whenever A is a coherent quantale (see Lemma 2.1).

In Section 3 we define the mz^n-elements in an algebraic quantale A (n is an arbitrary positive number). Since most of the proofs involve Lemma 2.1 we develop the theory in the framework of coherent frames. Denote by $z^n(A)$ the set of mz^n-elements of the quantale A and we obtain an ascending chain $(z^n(A))_{n\geq 1}$. An element of the union $\bigcup_{n=1}^{\infty} z^n(A)$ is called a higher order mz-element. The most important results of this section emphasize some properties of the chain $(z^n(A))_{n\geq 1}$. If the chain is stationary then the quantale A is said to be mz-terminating. We prove that any semiprime and hyperarchimedean quantale is mz-terminating.

Section 4 concerns the way in which the coherent quantale morphisms and their right adjoints preserve the higher order elements. Sufficient conditions are found for a coherent morphism to preserve the m-terminating property.

The paper ends with a section of final remarks.

2 Preliminaries on Quantales

In this section we shall recall some basic definitions and elementary results in the quantale theory (cf. [10, 24, 26]).

Let R be a unital commutative ring. Then $Id(R)$ can be endowed with a structure of multiplicative complete lattice with respect to the following operations:

- the meet of a family $(I_t)_{t\in T}$ of ideals of R is the intersection $\bigcap_{t\in T} I_t$;
- the join of a family $(I_t)_{t\in T}$ of ideals of R is the sum $\sum_{t\in T} I_t$;
- the multiplication of two ideals I and J is their product IJ.

This concrete algebraic structure of $Id(R)$ is one of the prototypes for the abstract notion of quantale. Then a *quantale* is a complete multiplicative lattice $(A, \bigvee, \wedge, \cdot, 0, 1)$, where the multiplication $\cdot$ is associative and for all $S \subseteq A$ and $a \in A$ the following infinite distributive law holds: $a \cdot (\bigvee S) = \bigvee\{a \cdot s \mid s \in S\}$ and $(\bigvee S) \cdot a = \bigvee\{s \cdot a \mid s \in S\}$. Usually, we shall write ab instead of $a \cdot b$ and the quantale $(A, \bigvee, \wedge, \cdot, 0, 1)$ is denoted by A.

The quantale A is said to be

- *integral*, if the structure $(A, \cdot, 1)$ is a monoid;
- *commutative*, if the multiplication $\cdot$ is commutative.

A *frame* is a quantale in which the multiplication coincides with the meet (see [15, 25]).

Throughout this paper, the quantales are assumed to be integral and commutative.

An element $c \in A$ is said to be *compact* if for any $S \subseteq A$ such that $c \leq \bigvee S$, there exists a finite subset S_0 of S such that $c \leq \bigvee S_0$. The set of compact elements of A is denoted by $K(A)$. The finitely generated ideals of the commutative ring R are the compact elements of the quantale $Id(R)$.

The quantale A is *algebraic* if any element $a \in A$ has the form $a = \bigvee X$ for some subset X of $K(A)$. It follows that any element a of an algebraic quantale can be written $a = \bigvee\{c \in K(A) \mid c \leq a\}$. An algebraic quantale A is said to be *coherent* if the top element 1 is a compact element and the set $K(A)$ of compact elements is closed under the multiplication. An example of a coherent quantale is $Id(R)$.

Recall from [26] that on each quantale A we can define:

- a binary operation $\rightarrow$, named *residuation* or *implication*: for all $a, b \in A$,

$$a \rightarrow b = \bigvee\{x \in A \mid ax \leq b\};$$

- a unary operation, named *annihilator operation*: for each $a \in A$,

$$a^{\perp} = a^{\perp_A} = a \rightarrow 0 = \bigvee\{x \in A \mid ax = 0\}.$$

Following a tradition in ring theory some authors use the notation $b : a$ for $a \rightarrow b$.

Recall from [26] that the implication $\rightarrow$ fulfills the following residuation rule: for all $a, b, c \in A$, $a \leq b \rightarrow c$ if and only if $ab \leq c$. Thus $(A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ is a (commutative) residuated lattice. In this paper we shall use without mention some elementary arithmetical properties of residuated lattices (the standard text on residuated lattices is the monograph [23]).

Following [3], an ideal P of a commutative ring R is a *prime ideal* if for all ideals I_1, I_2 of R, $I_1 I_2 \subseteq P$ implies $I_1 \subseteq P$ or $I_2 \subseteq P$. The notion of prime ideal is generalized to quantale theory : an element $p < 1$ of a quantale A is *m-prime* if for all $a, b \in A$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. If A is an algebraic quantale, then an element $p < 1$ is m-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. An element $m < 1$ is said to be *maximal* if for any $x \in A$ such that $m \leq x < 1$ we have $m = x$. Recall from [26] that $Spec(A)$ denotes the set of m-prime elements of A and $Max(A)$ denotes the set of maximal elements of A. If 1 is a compact element of the quantale A then for any $a < 1$ there exists $m \in Max(A)$ such that

$a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$. We remark that the set $Spec(R)$ of prime ideals in a commutative ring R is the prime spectrum of the quantale $Id(R)$ and the set $Spec_{Id}(L)$ of prime ideals in a bounded distributive lattice L is the prime spectrum of the frame $Id(L)$. Keeping the usual terminology, we say that $Spec(A)$ is the m-prime spectrum of A and $Max(A)$ is the maximal spectrum of A.

An m-prime element of the quantale A is *minimal* over $a \in A$ if for any m-prime element q, $a \leq q \leq p$ implies $q = p$. For any $a \in A$ there exists $p \in Spec(A)$ which is minimal over A.

If R is a ring, then its *Jacobson radical* is the ideal $J(A) = \bigcap Max(A)$ (cf. [3]). This notion can be generalized to a quantale A: $r(A) = \bigwedge Max(A)$ is the *Jacobson radical* of A (cf. [6]).

Let $B(A)$ be the Boolean algebra of complemented elements of A. Following [9], the quantale A is said to be *hyperarchimedean* if for any element $c \in K(A)$ there exists an integer $n \geq 1$ such that $c^n \in B(A)$.

Following [26] we recall that the *radical* $\rho(a)$ of an element a of the quantale A is defined by

$$\rho(a) = \rho_A(a) = \bigwedge\{p \in Spec(A) | a \leq p\}.$$

We remark that this notion is an abstraction of the radical of an ideal in a commutative ring. If $a = \rho(a)$ then a is said to be a *radical element* of A. The set of radical elements of A is denoted by $R(A)$. The quantale A is said to be *semiprime* if $\rho(0) = 0$.

We recall the following useful properties of the map $\rho : A \to A$. For all elements $a, b \in A$ the following hold

(2.1) $a \leq \rho(a)$;

(2.2) $\rho(ab) = \rho(a) \wedge \rho(b)$;

(2.3) $\rho(\rho(a)) = \rho(a)$;

(2.4) $\rho(a \vee b) = \rho(\rho(a) \vee \rho(b))$;

(2.5) $\rho(a) = 1$ if and only if $a = 1$;

(2.6) $\rho(a) \vee \rho(b) = 1$ if and only if $a \vee b = 1$;

(2.7) for all integers $n \geq 1$, $\rho(a^n) = \rho(a)$.

If S is a subset of A then the following equality holds:

(2.8) $\rho(\bigvee S) = \rho(\bigvee\{\rho(s)|s \in S\})$.

It is obvious that the set $R(A)$ is closed under the arbitrary meets, so it is a complete lattice. For any subset S of $R(A)$ we denote $\dot{\bigvee} S = \rho(\bigvee S)$. Then $(R(A), \dot{\bigvee}, \wedge, 0, 1)$ is a frame (see [26]). According to [6], if A is a coherent quantale then $K(R(A)) = \rho(K(A))$ and $R(A)$ is a coherent frame.

The following useful lemma is a quantale version of a well-known result in ring theory (see Proposition 1.8 of [3]).

Lemma 2.1 *[19] Let A be a coherent quantale and $a \in A$. Then the following hold:*

- *(1) $\rho(a) = \bigvee\{c \in K(A) \mid c^k \leq a$ for some integer $k \geq 1\}$;*
- *(2) For any $c \in K(A), c \leq \rho(a)$ if and only if $c^k \leq a$ for some integer $k \geq 1$;*
- *(3) A is semiprime if and only if for any integer $k \geq 1$, $c^k = 0$ implies $c = 0$.*

Let us fix an element a of a coherent quantale A. The interval $[a)_A = \{x \in A \mid a \leq x\}$ is closed under arbitrary joins of A. For all $x, y \in [a)_A$, we denote $x \cdot_a y = xy \vee a$. Then $[a)_A$ is closed under $\cdot_a$ and the algebraic structure $([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$ is a coherent quantale.

Let us consider two quantales A and B. Recall from [26] that a map $u : A \to B$ is a *quantale morphism* if it preserves the arbitrary joins and the multiplication (in particular, $u(0) = 0$); if $u(1) = 1$ then the quantale morphism u is an *integral quantale morphism*. We say that the quantale morphism u preserves the compact elements if $u(K(A)) \subseteq K(B)$.

Assume that A and B are coherent quantales. An integral quantale morphism $u : A \to B$ is said to be a *coherent quantale morphism* if it preserves the compact elements. The category of compact quantales and coherent quantale morphisms will be denoted CohQuant.

Any coherent quantale morphism $u : A \to B$ has a right adjoint $u_\star : B \to A$, defined by $u_\star(b) = \bigvee\{a \in A \mid u(a) \leq b\}$, for any $b \in B$.

3 mz^n-elements in Coherent Quantales

The definitions and the properties in this section have as starting point some notions and results of [9] and [17]. We begin this section by recalling the definitions of z-ideals and mz-ideals in ring theory.

Let R be a commutative (unital) ring. If $a \in R$ and F is a finite subset of R then we shall denote $\zeta_R(a) = \{M \in Max(A) \mid a \in M\}$, $Z_R(a) = \bigcap \zeta_R(a)$, $\zeta_R(F) = \{M \in Max(A) \mid F \subseteq M\}$ and $Z_R(F) = \bigcap \zeta_R(F)$ (cf. [14]).

An ideal I of R is said to be

- a *z-ideal* if for all $a, b \in A$, $\zeta_R(a) = \zeta_R(b)$ and $a \in I$ imply $b \in I$ (cf. [21]);
- an *mz-ideal* if for any finite subset F of R and $a \in A$, $\zeta_R(F) \subseteq \zeta_R(a)$ and $F \subseteq I$ imply $a \in I$ (cf. Definition 4.1 of [14]).

Let G be a finite subset of R and $< G >$ the ideal generated by G. Then we have $\zeta_R(< G >) = \zeta_R(F) = \bigcap_{a \in G} \zeta_R(a)$. Therefore, by applying Lemma 4.3 of [14], we can prove that the ideal I is an mz-ideal if and only if for all finitely generated ideals F, G of R, $\zeta_R(F) = \zeta_R(G)$ and $F \subseteq I$ imply $G \subseteq I$. It is clear that any mz-ideal of R is a z-ideal, but it is unknown if any z-ideal is an mz-ideal (cf. Example 4.2 of [14]).

Starting from these facts, the notion of mz-ideal was generalized in [13] to algebraic quantales.

Let us consider an algebraic quantale A. For any compact element c of the quantale A, we denote $\zeta_A(c) = \{m \in Max(A) | c \leq m\}$ and $z_A(c) = \bigwedge \zeta_A(c)$.

Lemma 3.1 *[13] If c, d are two compact elements of the algebraic quantale A then $\zeta_A(cd) = \zeta_A(c) \cup \zeta_A(d)$ and $\zeta_A(c \vee d) = \zeta_A(c) \cap \zeta_A(d)$. In particular, $\zeta_A(c^n) = \zeta_A(c)$ for each integer $n \geq 1$.*

Definition 3.2 *[13] An element a of the quantale A is an mz-element of A if for all compact elements c, d of A, $\zeta_A(c) = \zeta_A(d)$ and $c \leq a$ imply $d \leq a$.*

We denote by $z(A)$ the set of mz-elements of A. According to [13], $z(A)$ is closed under arbitrary meets and $Max(A) \subseteq z(A) \subseteq R(A)$.

In [9], Dube and Ighedo introduced the notion of z^n-ideal in a commutative ring, where n is a positive integer. In order to generalize this notion to quantales we shall recall the definition of a z^n-ideal. Let R be a commutative ring and n a positive integer. For each element a of R we denote $\zeta_R^n(a) = \{x^n | x \in Z_R(a)\}$. An ideal I of R is said to be a *z^n-ideal* if for all $a, b \in A$, $\zeta_R(a) = \zeta_R(b)$ and $a^n \in I$ imply $b^n \in I$. In accordance with Proposition 1 of [9], an ideal I of R is a z^n-ideal if and only if for each $a \in R$, $\zeta_R^n(a) \subseteq I$.

Definition 3.3 *Let n be a positive integer. An element a of an algebraic quantale A is said to be an mz^n-element of A if for all compact elements c, d of A, $\zeta_A(c) = \zeta_A(d)$ and $c^n \leq a$ imply $d^n \leq a$.*

The set of mz^n-elements of A will be denoted by $z^n(A)$. The mz^n-elements of the quantale $Id(R)$ of ideals in a commutative ring R will be called mz^n-ideals of R. Any z^n-ideal of R is an mz^n-ideal, but we don't know if the converse assertion is true. If A is an algebraic frame then the mz^n-elements of A coincide with the mz-elements of A.

For any compact element c of the algebraic quantale A and for any positive integer n we denote

$$z_A^n(c) = \bigvee\{x^n \mid x \in K(A), x \leq z_A(c)\}.$$

For any $c \in K(A)$ we have

$$z_A^1(c) = \bigvee\{x \mid x \in K(A), x \leq z_A(c)\} = z_A(c)$$

(because the quantale A is algebraic).

Lemma 3.4 *Let A be an algebraic quantale, $c \in K(A)$ and n a positive integer. Then for each $y \in K(A)$, the following equivalence holds: $y \leq z_A^n(c)$ if and only if $y \leq x^n$, for some compact element x of A such that $x \leq z_A(c)$.*

Proof: Assume that $y \in K(A)$ and $y \leq z_A^n(c)$, so there exist a positive integer k and the compact elements $x_1, \ldots, x_k$ such that $y \leq \bigvee_{i=1}^k x_i^n$ and $x_i \leq z_A(c)$, for all $i = 1, \cdots, k$. Denote $x = \bigvee_{i=1}^k x_i$, hence $x \in K(A)$ and $x \leq z_A(c)$. We also remark that $y \leq \bigvee_{i=1}^k x_i^n \leq (\bigvee_{i=1}^k x_i)^n = x^n$. The converse implication is obvious. □

Lemma 3.5 *Let A be an algebraic quantale, $c \in K(A)$ and m, n two positive integers. If $m \leq n$ then $z_A^n(c) \leq z_A^m(c)$.*

Proof: Let $m \leq n$ and $x \in K(A)$ then $x^n \leq x^m$. Therefore

$$z_A^n(c) = \bigvee\{x^n \mid x \in K(A), x \leq z_A(c)\}$$
$$\leq \bigvee\{x^m \mid x \in K(A), x \leq z_A(c)\} = z_A^m(c).$$

□

Let us fix a coherent quantale A for the rest of the section.

Lemma 3.6 *Let k be an arbitrary positive integer. For any $c \in K(A)$, the following hold:*

(1) $z_A(c^k) = z_A(c)$;

(2) $z_A^n(c^k) = z_A^n(c)$;

(3) $c^n \leq z_A^n(c)$.

Proof:

(1) By Lemma 3.1 we have $\zeta_A(c^k) = \zeta_A(c)$, so $z_A(c^k) = \bigwedge \zeta_A(c^k) = \bigwedge \zeta_A(c) = z_A(c)$.

(2) Using (1) the following equalities hold:

$$z_A^n(c^k) = \bigvee\{x^n \mid x \in K(A), x \leq z_A(c^k)\}$$
$$= \bigvee\{x^n \mid x \in K(A), x \leq z_A(c)\} = z_A^n(c).$$

(3) We remark that $c \leq \bigwedge\{m \in Max(A) \mid c \leq m\} = z_A(c)$, so $c^n \leq z_A^n(c)$. □

The following proposition extends Lemma 4.4 of [13] to the quantale framework.

Proposition 3.7 *For any $a \in A$, the following are equivalent*

(1) a is an mz^n-element of A;

(2) For all $c, d \in K(A)$, $\zeta_A(c) \subseteq \zeta_A(d)$ and $c^n \leq a$ imply $d^n \leq a$;

(3) For any $c \in K(A)$, $c \leq a$ implies $z_A^n(c) \leq a$.

Proof:

(1) ⇒ (2) Suppose that a is an mz-element of A. In order to prove the property (2), let c, d be two compact elements of A such that $\zeta_A(c) \subseteq \zeta_A(d)$ and $c^n \leq a$. By Lemma 3.1 we have

$$\zeta_A(c^n d) = \zeta_A(cd) = \zeta_A(c) \cup \zeta_A(d) = \zeta_A(d).$$

We remark that $c^n d \in K(A)$ (because A is a coherent quantale) and $c^n d \leq c^n \leq a$, therefore $d^n \leq a$ (because a is an mz^n-element of A).

(2) $\Rightarrow$ (3) Assume that c is a compact element of A such that $c \leq a$. We have to prove that $z_A^n(c) \leq a$. Suppose that x is a compact element of A such that $x \leq z_A(c)$. Then $x \leq m$ for any $m \in Max(A)$ such that $c \leq m$, therefore $\zeta_A(c) \subseteq \zeta_A(x)$. We remark that $c^n \leq c \leq a$, hence $x^n \leq a$ (according to the hypothesis (2)). It follows that $z_A^n(c) = \bigvee\{x^n \mid x \in K(A), x \leq z_A(c)\} \leq a$.

(3) $\Rightarrow$ (1) In order to prove that a is an mz^n-element of A, let c, d be two compact elements of A such that $\zeta_A(c) = \zeta_A(d)$ and $c^n \leq a$. We have to show that $d^n \leq a$. We remark that $z_A(c) = \bigwedge \zeta_A(c) = \bigwedge \zeta_A(d) = z_A(d)$, hence we get

$$z_A^n(c) = \bigvee\{x^n \mid x \in K(A), x \leq z_A(c)\}$$

$$= \bigvee\{x^n \mid x \in K(A), x \leq z_A(d)\} = z_A^n(d).$$

By virtue of hypothesis (3), $c^n \leq a$ implies $z_A^n(c^n) \leq a$. Therefore, by applying Lemma 3.6,(3) and (2), the following hold:

$$d^n \leq z_A^n(d) = z_A^n(c) = z_A^n(c^n) \leq a.$$

Then a is an mz^n-element of A. □

We note that Proposition 3.7 has a similar form with Proposition 1 of [9], but it does not generalize this. If we apply Proposition 3.7 to the particular quantale $Id(R)$ of ideals in a commutative ring R then we obtain a characterization of the mz^n-ideals of R, and not of the z^n-ideals of R.

Recall from [9] that an ideal I of a commutative ring R is a $\sqrt{z}$*-ideal* if $\sqrt{I}$ is a z-ideal. We can define a similar concept for quantale theory by replacing the z-ideals by mz-ideals: the ideal I will be called a $\sqrt{mz}$*-ideal* if $\sqrt{I}$ is an mz-ideal. We generalize this last notion to the quantale framework: an element a of the algebraic quantale A is said to be a $\rho(mz)$*-element* if $\rho(a)$ is an mz-element. The set of $\rho(mz)$-elements of A will be denoted by $z^\rho(A)$.

Proposition 3.8 *For any positive integer n, the following hold*

(1) $z^n(A) \subseteq z^{n+1}(A)$*;*

(2) $z^n(A) \subseteq z^\rho(A)$*;*

(3) $R(A) \cap z^n(A) = z(A)$.

Proof:

(1) Assume that a is an element of $z^n(A)$. We have to prove that $a \in z^{n+1}(A)$. Let c be a compact element of A such that $c \leq a$. By Proposition 3.7 we have $z_A^n(c) \leq a$. Using Lemma 3.5 we get $z_A^{n+1}(c) \leq z_A^n(c)$, hence $z_A^{n+1}(c) \leq a$. A new application of Proposition 3.7 gives $a \in z^{n+1}(A)$.

(2) Let a be an element of $z^n(A)$. We have to prove that $\rho(a)$ is an mz-element. Let c, d be two compact elements of A such that $\zeta_A(c) = \zeta_A(d)$ and $c^n \leq \rho(a)$. Then there exists a positive integer k such that $c^k \leq a$ (by Lemma 2.1(2)), hence $c^{kn} \leq a$. According to Lemma 3.1, $\zeta_A(c^k) = \zeta_A(c)$, hence $\zeta_A(c^k) = \zeta_A(d)$. But $a \in z^n(A)$, so $c^k \in K(A)$, $\zeta_A(c^k) = \zeta_A(d)$ and $(c^k)^n \leq a$ imply $d^n \leq a$, therefore $d \leq \rho(a)$ (by Lemma 2.1(2)). We proved that $\rho(a)$ is an mz-element of A.

(3) We know that $z(A) \subseteq R(A)$ (cf. Lemma 4.3 of [13]) and $z(A) \subseteq z^n(A)$ (cf. Proposition 3.7(1)), so $z(A) \subseteq R(A) \cap z^n(A)$. In order to verify the converse inclusion assume that $a \in R(A) \cap z^n(A)$. We have to show that a is an mz-element. Assume that c, d are two compact elements such that $\zeta_A(c) = \zeta_A(d)$ and $c \leq a$. Thus $c^n \leq a$, so $d^n \leq a$ (because a is an mz^n-element). Using Lemma 2.1(2) we obtain $d \leq \rho(a)$, therefore $d \leq a$ (because $a \in R(A)$). We conclude that $a \in z(A)$, so $R(A) \cap z^n(A) \subseteq z(A)$. □

Corollary 3.9 *(1)* $z(A) \subseteq z^2(A) \subseteq \cdots \subseteq z^n(A) \subseteq z^{n+1}(A) \subseteq \cdots$;

(2) $\bigcup_{n=1}^{\infty} z^n(A) \subseteq z^{\rho}(A)$.

Proof:

(1) By Propostion 3.8(1).

(2) By Proposition 3.8(2). □

Adapting the terminology of [9], the ascending chain $z(A) \subseteq z^2(A) \subseteq z^3(A) \subseteq \cdots$ from Corollary 3.9(1) will be called the $\imath mz$-tower of the coherent quantale A. An element of the set $z^{\infty}(A) = \bigcup_{n=1}^{\infty} z^n(A)$ will be called a *higher order mz-element* of A.

Corollary 3.10 *(1) Any higher order mz-element of A is a $\rho(mz)$-element;*

(2) $R(A)\bigcap z^\infty(A) = z(A)$.

Proof:

(1) By Corollary 3.9(2).

(2) By Proposition 3.8(3), $R(A)\bigcap z^\infty(A) = \bigcup_{n=1}^\infty (R(A)\cap z^n(A)) = z(A)$. □

Proposition 3.11 *$z^\infty(A)$ is closed under the multiplication of A.*

Proof: Let a and b be two elements of $z^\infty(A)$, so $a \in z^m(A)$ and $b \in z^n(A)$, for some positive integers $m \geq 2$ and $n \geq 2$ (by using Proposition 3.8(1)). We want to prove that $ab \in z^{m+n}(A)$. Let c, d be two compact elements of A such that $\zeta_A(c) = \zeta_A(d)$ and $c^{m+n} \leq ab$. Since $m + n \leq mn$ we have $(c^m)^n = c^{mn} \leq c^{m+n} \leq ab \leq b$. We remark that $\zeta_A(c^m) = \zeta_A(c) = \zeta_A(d)$, $c^m, d \in K(A)$ and $(c^m)^n \leq b$, therefore $d^n \leq b$ (because $b \in z^n(A)$). In a similar way we obtain $d^m \leq a$, therefore $d^{m+n} \leq ab$. □

Proposition 3.12 *If $a \in z^n(A)$ then $\rho(a) = \bigvee\{c \in K(A) \mid c^n \leq a\}$.*

Proof: The inequality $\bigvee\{c \in K(A) \mid c^n \leq a\} \leq \rho(a)$ is ensured by Lemma 2.1(1). In order to prove that $\rho(a) \leq \bigvee\{c \in K(A) \mid c^n \leq a\}$ assume that d is a compact element of A such that $d \leq \rho(a)$. By Lemma 2.1(2) there exists a positive integer k such that $d^k \leq a$. Then $d^k \in K(A)$, $\zeta_A(d^k) = \zeta_A(d)$ and $(d^k)^n \leq d^k \leq a$, so $d^n \leq a$ (because $a \in z^n(A)$). It follows that $\rho(a) \leq \bigvee\{c \in K(A) \mid c^n \leq a\}$, so $\rho(a) = \bigvee\{c \in K(A) \mid c^n \leq a\}$. □

Proposition 3.13 *If $a \in z^n(A)$ and $c \in K(A)$ then $c^n \to a \in z^n(A)$.*

Proof: Assume that a is an mz^n-element of A. Let x, y be two compact elements of A such that $\zeta_A(x) = \zeta_A(y)$ and $x^n \leq c^n \to a$. Then xc and yc are compact elements of A, $(xc)^n \leq a$ and $\zeta_A(xc) = \zeta_A(x) \cup \zeta_A(c) = \zeta_A(y) \cup \zeta_A(c) = \zeta_A(yc)$. It follows that $(yc)^n \leq a$, hence $y^n \leq c^n \to a$. Therefore $c^n \to a$ is an mz^n-element of A. □

Proposition 3.14 *Assume that $a \in z^n(A)$, $c \in K(A)$ and $m \geq n$. Then $\rho(c^m \to a) = c \to \rho(a)$.*

Proof: By virtue of Proposition 3.8(1), $a \in z^n(A)$ and $m \geq n$ imply $a \in z^m(A)$. According to Proposition 3.13, $c^m \to a \in z^m(A)$. Applying Proposition 3.12 and the residuation rule the following equalities hold:

$$\begin{aligned} \rho(c^m \to a) &= \bigvee\{x \in K(A) \mid x^m \leq c^m \to a\} \\ &= \bigvee\{x \in K(A) \mid (xc)^m \leq a\} \\ &= \bigvee\{x \in K(A) \mid xc \leq \rho(a)\} \\ &= \bigvee\{x \in K(A) \mid x \leq c \to \rho(a)\} \\ &= c \to \rho(a). \end{aligned}$$ □

Remark 3.15 *Assume that $a \in z^n(A)$, $c \in K(A)$ and $m \geq n$. According to the previous proposition, the sequence $(\rho(c^m \to a))_{m \geq 1}$ is stationary.*

Lemma 3.16 *For any compact element c of A we have $\zeta_A(c) = \zeta_{R(A)}(\rho(c))$.*

Proof: We know from [6] that $Max(R(A)) = Max(A)$, therefore

$$\begin{aligned} \zeta_A(c) &= \{m \in Max(A) \mid c \leq m\} \\ &= \{m \in Max(R(A)) \mid \rho(c) \leq m\} = \zeta_{R(A)}(\rho(c)). \end{aligned}$$ □

Corollary 3.17 *For all compact elements c, d of A, $\zeta_{R(A)}(\rho(c)) = \zeta_{R(A)}(\rho(d))$ if and only if $\zeta_A(c) = \zeta_A(d)$.*

The following proposition shows that the mz-elements of the frame $R(A)$ coincide with the mz-elements of the quantale A. Recall that $K(R(A)) = \{\rho(c) \mid c \in K(A)\}$.

Proposition 3.18 $z(R(A)) = z(A)$.

Proof: Firstly, we shall prove that $z(R(A)) \subseteq z(A)$. Assume that $a \in z(R(A))$. Let c, d be two compact elements of A such that $\zeta_A(c) = \zeta_A(d)$ and $c \leq a$. Thus $\rho(c), \rho(d)$ are compact elements of $R(A)$ and $\rho(c) \leq \rho(a) = a$. By Corollary 3.17 we have $\zeta_{R(A)}(\rho(c)) = \zeta_{R(A)}(\rho(d))$, hence $\rho(d) \leq a$. Then $d \leq \rho(d) \leq a$, hence $a \in z(A)$.

In order to prove the converse inclusion $z(A) \subseteq z(R(A))$ consider an element a of $z(A)$. We have to prove that $a \in z(R(A))$. Let x, y be two compact elements of $R(A)$ such that $\zeta_{R(A)}(x) = \zeta_{R(A)}(y)$ and $x \leq a$. One can find $c, d \in K(A)$ such that $x = \rho(c), y = \rho(d)$, so $\zeta_A(c) = \zeta_A(d)$ (by Corollary 3.17) and $c \leq \rho(c) \leq \rho(a) = a$. Then $d \leq a$, hence $y = \rho(d) \leq \rho(a) = a$. $\Box$

Proposition 3.19 *Assume that $p \in Spec(A)$. If p is minimal over a higher order mz-element a of A then $p \in z(A)$.*

Proof: Let a be a higher order mz-element of A, so there exists a positive integer n such that $a \in z^n(A)$. By Proposition 3.8(2) we get $a \in z^\rho(A)$, i.e. $\rho(a)$ is an mz-element. Let q be an m-prime element of A such that $\rho(a) \leq q \leq p$, so $a \leq q \leq p$, hence $p = q$ (because p is a minimal over a). Thus p is a minimal over $\rho(a)$. In accordance with Proposition 6.11 of [13], p is an mz-element of A. $\Box$

The quantale A is *mz-terminating* if there exists a positive integer n such that $z^n(A) = z^{n+k}(A)$, for any integer k.

Proposition 3.20 *Any semiprime and hyperarchimedean quantale A is mz-terminating.*

Proof: According to Theorem 6.2 of [13], the semiprime and hyperarchimedean A is a V-quantale, i.e. any element of A is a strong z-element. Then any element of A is an mz-element, hence $z(A) = A$. By virtue of Proposition 3.8(1), $z(A) = z^2(A) = z^3(A) = \cdots = A$. $\Box$

By virtue to proof of Proposition 3.20, the mz^n-elements of a semiprime and hyperarchimedean quantale A coincide with the mz-elements. It would be interesting to know if this property remains valid for any hyperarchimedean quantale.

The quantale A is said to be *radically mz-covered* if $z^\infty(A) = z^\rho(A)$. By Corollary 3.9(2), a coherent quantale A is radically mz-covered if and only if $z^\rho(A) \subseteq z^\infty(A)$. We say that a commutative ring R is *radically mz-covered* if the quantale $Id(R)$ is radically mz-covered, i.e. any $\sqrt{mz}$-ideal is a higher order mz-ideal.

Following [3], an ideal I of a commutative ring R is an *SFT-ideal* if there exists a finitely generated ideal $F \subseteq I$ and a positive integer k such that $x^k \in F$, for any $x \in I$. R is said to be an *SFT-ring* if any ideal of R is an SFT-ideal.

Taking into account the previous notions, we say that:

- an element a of a coherent quantale A is an *SFT-element* if there exist a compact element $c \leq a$ and a positive integer k such that for each $y \in K(A)$, $y \leq a$ implies $y^k \leq c$;
- the coherent quantale A is said to be an *SFT-quantale* if any element $a \in A$ is an SFT-element.

In what follows we will prove that any SFT-quantale is radically mz-covered. In order to obtain this result we need the following lemma, which is a quantale version of Proposition 2.1(iv) of [2].

Lemma 3.21 *Let A be an SFT-quantale. Then for any $a \in A$ there exists a positive integer k such that for each $y \in K(A)$, $y \leq \rho(a)$ implies $y^k \leq a$.*

Proof: Assume that $a \in A$. Since $\rho(a)$ is an SFT-element of A there exist a positive integer s and a compact element $c \leq \rho(a)$ such that the following implication holds:

(i) $x \in K(A)$ and $x \leq \rho(a)$ imply $x^s \leq c$.

Applying Lemma 2.1(2) to the inequality $c \leq \rho(a)$ one can find a positive integer t such that $c^t \leq a$. We take $k = st$. Let $y \in K(A)$ such that $y \leq \rho(a)$. Using (i), from $y \in K(A)$ and $y \leq \rho(a)$ we get $y^s \leq c$, therefore $y^k = y^{st} \leq c^t \leq a$. □

Theorem 3.22 *Any SFT-quantale A is radically mz-covered.*

Proof: Let a be an element of $z^{\rho}(A)$, hence $\rho(a) \in z(A)$. By virtue of Lemma 3.21, there exists a positive integer k such that the following implication holds:

(a) $y \in K(A)$ and $y \leq \rho(a)$ imply $y^k \leq a$.

We have to prove that $a \in z^k(A)$. Let c, d be two compact elements of A such that $\zeta_A(c) = \zeta_A(d)$ and $c^k \leq a$. In accordance with Lemma 2.1(2), we have $c \leq \rho(a)$, hence $d \leq \rho(a)$ (because $\rho(a) \in z(A)$). According to (a), from $d \in K(A)$ and $d \leq \rho(a)$ we get $d^k \leq a$, hence $a \in z^k(A)$. □

Remark 3.23 *Let R be an SFT-ring. Applying Theorem 3.22 to the SFT-quantale $Id(R)$ it follows that the ring R is radically mz-covered.*

Remark 3.24 *Let Z be the ring of integers. We know that any ideal of Z is principal, hence the mz^n-ideals and z^n-ideals of Z coincide. Let n be an arbitrary integer and $I =< 2^{n+1} >$ the principal ideal of Z generated by 2^{n+1}. Using Example 5 of [9], we conclude that I is an mz^{n+1}-ideal of Z, but not an mz^n-ideal. Then the ring Z is not mz-terminating.*

4 mz^n-elements and the Quantale Morphisms

In this section we define the strict quantale morphisms as a quantale abstraction of the strict ring morphisms (see [9] for this last notion).

Necessary and sufficient conditions are established for the right adjoint of a coherent quantale morphism to preserve the mz^n-elements (see Theorem 4.7). This result is used to determine sufficient conditions for a strict quantale morphism to preserve the mz-terminating property.

Let $u : A \to B$ be a coherent quantale morphism. The following two lemmas present some elementary facts on the adjoint pair $(u, u_\star)$.

Lemma 4.1 *The following properties fulfill*

(1) (the adjointness property): For all $a, b \in A$, $u(a) \leq b$ *if and only if* $a \leq u_\star(b)$;

(2) For any $a \in A$, $a \leq u_\star(u(a))$;

(3) For any $b \in B$, $u(u_\star(b)) \leq b$;

(4) For any $b \in B$, $u_\star(b) = 1$ *if and only if* $b = 1$.

Lemma 4.2 *If* u *is surjective then* $u(K(A)) = K(B)$.

The following proposition contains a quantale generalization of Lemma 2 of [9].

Proposition 4.3 *Let* A, B *be two coherent quantales and* $u : A \to B$ *a surjective coherent quantale morphism. Then the following assertions are equivalent:*

(1) For any a, $a < 1$ *implies* $u(a) < 1$;

(2) For any $m \in Max(A)$ *we have* $u(m) \in Max(B)$;

(3) For any $m \in Max(A)$ *we have* $u(m) < 1$;

(4) For any $m \in Max(A)$ *we have* $u_\star(u(m)) = m$;

(5) For all $c, d \in K(A)$, $\zeta_B(u(d)) \subseteq \zeta_B(u(c))$ *implies* $\zeta_A(d) \subseteq \zeta_A(c)$;

(6) For all $c, d \in K(A)$, $\zeta_B(u(d)) = \zeta_B(u(c))$ *implies* $\zeta_A(d) = \zeta_A(c)$.

Proof:

(1) ⇒ (2) Assume that $m \in Max(A)$, so $m < 1$. Applying hypothesis (1) we obtain $u(m) < 1$. Then $u(m) \leq n$, for some maximal element n of B, hence, using the adjointness property we get $m \leq u_\star(n) < 1$ (the last strict inequality follows from Lemma 4.1(4)). Since m is a maximal element of A we have $u_\star(n) = m$. By Lemma 4.1(2) we have $n \leq u(u_\star(n)) = u(m)$, so $n = u(m)$ (because $n \in Max(B)$ and $u(m) < 1$), therefore $u(m) \in Max(B)$.

(2) ⇒ (3) Obvious.

(3) ⇒ (4) Let m be a maximal element of A. Then $u(m) < 1$, hence $u_\star(u(m)) < 1$ (by Lemma 4.1(4)). Therefore, using Lemma 4.1(2) we get $m \leq u_\star(u(m)) < 1$, hence $m = u_\star(u(m))$.

(4) ⇒ (5) Let c, d be two compact elements of A such that $\zeta_B(u(d)) \subseteq \zeta_B(u(c))$. In order to show that $\zeta_A(d) \subseteq \zeta_A(c)$ assume that $m \in \zeta_A(d)$, so $d \leq m \in Max(A)$. From $m = u_*(u(m))$ we obtain $u(m) < 1$. It follows that $u(d) \leq u(m) \leq n$, for some maximal element n of B, so $n \in \zeta_B(u(d))$. Thus $n \in \zeta_B(u(c))$, so $u(c) \leq n$. By the adjointness property we get $c \leq u_\star(n)$. According to the hypothesis (4) we have $m = u_\star(u(m)) \leq u_\star(n) < 1$, so $m = u_\star(n)$ (because $m \in Max(A)$). We conclude that $c \leq m$, so $m \in \zeta_A(c)$. Then the inclusion $\zeta_A(d) \subseteq \zeta_A(c)$ follows.

(5) ⇒ (1) Let a be an element of A such that $a < 1$. Assume by absurdum that $u(a) = 1$. We know that $a = \bigvee\{c \in K(A) \mid c \leq a\}$, therefore $1 = u(a) = \bigvee\{u(a) \mid c \in K(A), c \leq a\}$ (since u preserves the joins). But $1 \in K(A)$, so there exist a positive integer m and the compact elements $c_1, \cdots, c_m$ such that $1 = \bigvee_{i=1}^m u(c_i) = u(\bigvee_{i=1}^m c_i)$ and $c_i \leq a$, for $i = 1, \ldots, m$. Denoting $c = \bigvee_{i=1}^m c_i$ we have $c \in K(A)$, $c \leq a$ and $u(c) = 1$, hence $\zeta_B(u(c)) = \emptyset$. Then $\zeta_B(u(c)) \subseteq \zeta_B(u(1))$, hence $\zeta_B(c) \subseteq \zeta_B(1) = \emptyset$. We conclude that $c = 1$, contradicting $u(c) \leq u(a) < 1$.

(5) ⇒ (6) Obvious.

(6) ⇒ (5) Let c, d be two compact elements of A such that $\zeta_B(u(d)) \subseteq \zeta_B(u(c))$. According to Lemma 3.1 we get

$$\zeta_B(u(c)) = \zeta_B(u(c)) \cup \zeta_B(u(d)) = \zeta_B(u(c)u(d)) = \zeta_B(u(cd)).$$

Due to hypothesis (6) we obtain $\zeta_A(c) = \zeta_A(cd)$, so $\zeta_A(c) = \zeta_A(c) \cup \zeta_A(d)$, hence $\zeta_A(d) \subseteq \zeta_A(c)$.

□

Let $u : A \to B$ be a surjective coherent quantale morphism. Following the terminology of [9], u is said to be a *strict quantale morphism* if it satisfies the equivalent properties of the previous proposition.

Lemma 4.4 *Assume that $u : A \to B$ is a strict quantale morphism, $b \in B$ and n is a positive integer. If $u_\star(b) \in z^n(A)$ then $b \in z^n(B)$.*

Proof: Assume that $u_\star(b) \in z^n(A)$. In order to show that $b \in z^n(B)$ let x, y be two compact elements of B such that $\zeta_B(x) = \zeta_B(y)$ and $x^n \leq b$. According to Lemma 4.2 there exist two compact elements c, d of A such that $x = u(c)$ and $y = u(d)$, hence $\zeta_B(u(c)) = \zeta_B(u(d))$. Since u is a strict quantale morphism one can apply Proposition 4.3(6), therefore $\zeta_A(c) = \zeta_A(d)$. We observe that $u(c^n) = x^n \leq b$, hence $c^n \leq u_\star(b)$ (by the adjointness property). By hypothesis, $u_\star(b)$ is an mz^n-element of A, so $\zeta_A(c) = \zeta_A(d)$ and $c^n \leq u_\star(b)$ imply $d^n \leq u_\star(b)$. Using the adjointness property we get $y^n = u(d^n) \leq b$. Therefore, we conclude that b is an mz^n-element of B. □

Lemma 4.5 *[12] If $q \in Spec(B)$ then $u_\star(q) \in Spec(A)$.*

Proposition 4.6 *Let $u : A \to B$ be a coherent quantale morphism and n be a positive integer. Then the following properties are equivalent:*

(1) For any $b \in z^n(B)$ we have $u_\star(b) \in z^n(A)$;

(2) For any $q \in Max(B)$ we have $u_\star(q) \in z^n(A)$.

Proof:

(1) ⇒ (2) Let q be a maximal element of B. By $Max(B) \subseteq z(B) \subseteq z^n(B)$, $q \in z^n(B)$ and so by hypothesis (1) we get $u_\star(b) \in z^n(A)$.

(2) ⇒ (1) Assume that $b \in z^n(B)$. We have to show that $u_\star(b) \in z^n(A)$. Let $c, d \in K(A)$ such that $\zeta_A(c) = \zeta_A(d)$ and $c^n \leq u_\star(b)$. Then $c^n, d^n \in K(A)$, $u(c), u(d) \in K(B)$ and $(u(c))^n = u(c^n) \leq b$ (the last inequality follows by adjointness property).

Now we will prove that $\zeta_B(u(c)) = \zeta_B(u(d))$. Let $q \in Max(B)$ such that $q \in \zeta_B(u(c))$, so $u(c) \leq q$. By the adjointness property we have $c \leq u_\star(q)$, so $c^n \leq u_\star(q)$. By virtue of hypothesis (2), $u_\star(q)$ is an mz^n-element of A, so $c^n \leq u_\star(q)$ implies $d^n \leq u_\star(q)$, therefore $(u(d))^n = u(d^n) \leq q$. It follows that $u(d) \leq q$ (because $q \in Spec(B)$), so $q \in \zeta_B(u(d))$. We have proven the inclusion $\zeta_B(u(c)) \subseteq \zeta_B(u(d))$. The converse inclusion $\zeta_A(u(d)) \subseteq \zeta_A(u(c))$ follows in a similar way, hence $\zeta_B(u(c)) = \zeta_B(u(d))$.

But b is an mz^n-element of B, so $u(c), u(d) \in K(B)$, $\zeta_A(u(c)) = \zeta_A(u(d))$ and $(u(c))^n \leq b$ imply $(u(d))^n \leq b$. Then $u(d^n) \leq b$, hence $d^n \leq u_\star(b)$ (by the adjointness property). We conclude that $u_\star(b) \in z^n(A)$.

□

Theorem 4.7 *Let $u : A \to B$ be a coherent quantale morphism and n a positive integer. Then the following properties are equivalent:*

(1) For any $b \in z^n(B)$ we have $u_\star(b) \in z^n(A)$;

(2) For any $q \in Max(B)$ we have $u_\star(q) \in z^n(A)$;

(3) For any $q \in Max(B)$ we have $u_\star(q) \in z(A)$;

(4) For any $b \in z(B)$ we have $u_\star(b) \in z(A)$.

Proof:

(1) ⇒ (2) By Proposition 4.6.

(3) ⇒ (4) We apply Proposition 4.6 for $n = 1$.

(2) ⇒ (3) Assume that $q \in Max(B)$, hence $q \in Spec(B)$. Using Lemma 4.5 we get $u_\star(q) \in Spec(A)$, so $u_\star(q) \in R(A)$. By hypothesis (2) we have $u_\star(q) \in z^n(A)$, hence $u_\star(q) \in R(A) \cap z^n(A) = z(A)$ (the last equality is exactly Proposition 3.8(3)).

(3) ⇒ (2) This implication follows using the inclusion $z(A) \subseteq z^n(A)$ (cf. Proposition 3.8(1)).

□

Theorem 4.8 *Let $u : A \to B$ be a strict quantale morphism. If $u_\star(Max(B)) \subseteq z(A)$ and the quantale A is mz-terminating then the quantale B is mz-terminating.*

Proof: Let n be a positive integer such that $z^n(A) = z^{n+i}(A)$, for each positive integer i. We want to prove that $z^n(B) = z^{n+i}(B)$, for each positive integer i. According to Proposition 3.8(1) it suffices to check that $z^{n+i}(B) \subseteq z^n(B)$.

Let us consider an arbitrary element y of $z^{n+i}(B)$. By virtue of hypothesis $u_\star(Max(B)) \subseteq z(A)$ the four properties of Theorem 4.7 are fulfilled, hence $u_\star(y) \in z^{n+i}(A) = z^n(A)$. But u is a strict quantale morphism, hence, by using Lemma 4.4, from $u_\star(y) \in z^n(A)$ we obtain $y \in z^n(B)$. We have proven that $z^{n+i}(B) \subseteq z^n(B)$, so $z^{n+i}(B) = z^n(B)$. Then the quantale B is mz-terminating. □

Let a be an element of the coherent quantale A. The map $u_a^A : A \to [a)_A$, defined by $u_a^A(x) = a \vee x$ for any $x \in A$, is a coherent quantale morphism. u_a^A is surjective, so $K([a)_A) = u_a^A(K(A))$ (cf. Lemma 4.2).

Lemma 4.9 $(u_a^A)_\star(x) = x$*, for any* $x \in [a)_A$.

Proof: Assume that $x \in [a)_A$. By virtue of the definition of $(u_a^A)_\star$ we get

$$\begin{aligned}(u_a^A)_\star(x) &= \bigvee\{b \in A \mid u_a^A(b) \leq x\}\\ &= \bigvee\{b \in A \mid a \vee b \leq x\} = \bigvee\{b \in A \mid b \leq x\} = x.\end{aligned}$$ □

Recall that the Jacobson radical of the quantale A is $r(A) = \bigwedge Max(A)$.

Lemma 4.10 u_a^A *is a strict quantale morphism if and only if* $a \leq r(A)$.

Proof: Assume that u_a^A is a strict quantale morphism. Let m be a maximal element of A, so $u_a^A(m) < 1$ (by Proposition 4.3(3)). Then $a \vee m < 1$, so we get $a \leq m$ (because $a \not\leq m$ iff $a \vee m = 1$). It follows that $a \leq \bigwedge Max(A) = r(A)$.

Conversely, suppose that $a \leq r(A)$, so $a \leq m$ for any $m \in Max(A)$. Then $m \in Max(A)$ implies that $u_a^A(m) = a \vee m = m < 1$. In accordance with Proposition 4.3(3) it follows that u_a^A is a strict quantale morphism. □

Proposition 4.11 *Assume that* $a \leq r(A)$*. If the coherent quantale* A *is* mz*-terminating then the quantale* $[a)_A$ *is* mz*-terminating.*

Proof: According to Lemma 4.10, u_a^A is a strict quantale morphism. It is obvious that $Max([a)_A) = \{m \in Max(A) | a \leq m\}$, so, by using Lemma 4.9 we get $(u_a^A)_\star(Max([a)_A)) \subseteq Max(A) \subseteq z(A)$. Therefore, by applying Theorem 4.8 to the strict quantale morphism u_a^A, it follows that the quantale $[a)_A$ is mz-terminating. □

Lemma 4.12 *If $u : A \to B$ is a coherent quantale morphism then $u_\star(\rho_B(b)) = \rho_A(u_\star(b))$, for any $b \in B$.*

Proof: Let b be an element of the quantale B and $c \in K(A)$. Using the adjointness property and Lemma 2.1(2) we get the following equivalences:

$$\begin{aligned} c \leq u_\star(\rho_B(b)) &\Leftrightarrow u(c) \leq \rho_B(b) \\ &\Leftrightarrow (u(c))^n \leq b, \text{ for some positive integer } n \\ &\Leftrightarrow c^n \leq u_\star(b), \text{ for some positive integer } n \\ &\Leftrightarrow c \leq \rho_A(u_\star(b)). \end{aligned}$$

We have proven that $c \leq u_\star(\rho_B(b)) \Leftrightarrow c \leq \rho_A(u_\star(b))$ for each compact element c of A, therefore $u_\star(\rho_B(b)) = \rho_A(u_\star(b))$. □

Theorem 4.13 *Let $u : A \to B$ be a strict quantale morphism. If the quantale A is radically mz-covered then the quantale B is radically mz-covered.*

Proof: Assume that $z^\infty(A) = z^\rho(A)$. Let us consider an arbitrary element b of $z^\rho(B)$, so $\rho_B(b)$ is an mz-element of B. In accordance with Theorem 4.7, $u_\star(\rho_B(b))$ is an mz-element of A. Using Lemma 4.12 it follows that $\rho_A(u_\star(b))$ is an mz-element of A, hence $u_\star(b) \in z^\rho(A)$. Thus $u_\star(b) \in z^\infty(A)$, so there exists a positive integer n such that $u_\star(b) \in z^n(A)$. Since u is a strict quantale morphism one can apply Lemma 4.4, so we obtain $b \in z^n(B)$, hence $b \in z^\infty(B)$. Thus $z^\rho(A) \subseteq z^\infty(A)$, so B is radically mz-covered. □

5 Final remarks

In this paper we studied the mz^n-elements of a coherent quantale A, a notion that generalizes the mz-elements of A (defined in [13] as an abstraction of the mz-ideals of a commutative ring [14]).

An important problem is to find a suitable abstract setting (quantales or other classes of multiplicative lattices [11]) in order to generalize the z-ideals [21] and the z^n-ideals [9]. To achieve this goal, we will have to start from the notion of the *principal element* of a multiplicative lattice, introduced by Dilworth in [7] and studied especially in [1].

Let A be a quantale. An element $e \in A$ is said to be *principal* if for all $a, b \in A$ the following two conditions are fulfilled:

- $a \wedge be = ((a : e) \wedge b)e$;

- $(a \vee be) : e = (a : e) \vee b$.

Let $Pr(A)$ be the set of principal elements of A. The following definition of a z-element of A is directly inspired by the notion of z-ideal of a commutative ring [21].

Definition 5.1 *An element a of the quantale A is a z-element of A if for all principal elements c, d of A, $\zeta_A(c) = \zeta_A(d)$ and $c \leq a$ imply $d \leq a$.*

Let I be an ideal of a commutative ring. Recall that $Id(R)$ is the quantale of ideals of R. We observe that I is a z-ideal in the sense of [21] if and only if it is a z-element of the quantale $Id(R)$. Then the notion of z-element, introduced by Definition 5.1, is an appropriate generalization of the z-ideals defined by Mason in [21].

The definition of the z^n-elements of A can be formulated in a similar way, following Definition 1 of [9].

The question arises what conditions must be met by the quantale A in order to be able to extend the properties of z-ideals and z^n -ideals of commutative rings to z-elements and to the z^n-elements of A.

D. D. Anderson introduced in [1] the *r-lattices*, a class of multiplicative lattices in which numerous properties related to the principal elements can be proved. In particular, any r-lattice A is a coherent quantale and $Pr(A) \subseteq K(A)$. It is not difficult to see that in an r-lattice A any mz-element is a z-element. We don't know if the converse implication holds (cf. Example 4.2 of [14], it is not known if there exists "an example of a z-ideal which is not an mz-ideal").

We think that the r-lattices constitute a good candidate for a rich theory of the z-elements and the z^n-elements.

Another notion of z-element in a multiplicative lattice was defined by Manjarekar and Chavan in [18]. Following [18], an element a of a quantale A is a *z-element* of A if for all elements c, d of A, $\zeta_A(c) = \zeta_A(d)$ and $c \leq a$ imply $d \leq a$.

For this definition of the notion of z-element to be an adequate generalization of Mason's z-ideals, the following two conditions should be equivalent for any ideal I of a commutative ring R:

(i) I is a z-ideal;

(ii) I is a z-element (in the sense of Manjarekar and Chavan) of the quantale $Id(R)$.

It is obvious that $(ii) \Rightarrow (i)$ (in fact, any z-element in the sense of Manjarekar and Chavan is an mz-element), but it is unknown if the converse implication fulfills (cf. Example 4.2 of [14]).

The recent paper [8] contains remarkable results regarding the z-elements in the sense of Manjarekar and Chavan. Among the open problems presented in [8] are the following two:

- the study of the z-elements in non-commutative quantales;
- the study of the z^n-elements in the framework of multiplicative lattices.

Similarly, the study of mz-elements in non-commutative quantale can be viewed as an open problem. It would also be interesting to investigate which of the results obtained in [8] for z- the elements of Manjarekar and Chavan remain valid in the case of mz-elements of a quantale.

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