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# On Sharp Bounds of Fault-Tolerant Partition Dimension of Convex Polytopes

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**Abstract**

Graph theory is a fundamental and powerful tool for designing and modeling networks. It plays a vital role in diverse real-world systems, including social, computer, biological, ecological, and neural networks. Convex polytopes are convex sets of elements contained in the Euclidean space $\mathbb{R}^n$. They arise in numerous domains such as linear programming, finance, computer science, electrical engineering, bioinformatics, and chemistry. From the graph-theoretic perspective, some sharp upper bounds for the partition dimension of convex polytopes have been established recently. Inspired by this, in this paper, we aim to investigate the fault-tolerant partition dimension (FTPD), which is an extension of the partition dimension. Here, we compute bounds for the FTPD of certain convex polytopes. These results contribute to a deeper understanding of the robustness of network structures modeled by convex polytopes and may support further applications in fault-tolerant system design.

**Keywords:** convex polytopes, partition dimension, fault-tolerant partition dimension



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## 1 Introduction and Preliminaries

Convex polytopes are basic geometric objects that include a wide range of mathematical problems and structures. Convex polytopes' fundamental theory addresses faces, normal cones, duality, and especially polarity. In systems biology, a convex polytope represents the flux space associated with a metabolic network. Important biological insights can be obtained by uniformly sampling the interior of the polytope [11]. In computational finance, the collection of all feasible stock market portfolios forms a convex polytope. Volume computation and uniform sampling of this set are useful techniques for identifying potential financial crises [7]. Numerous mathematical fields, including integration theory, algebraic geometry, algebraic topology, linear and combinatorial optimization, and system control, have benefited from the use of convex polytopes [10, 30]. Convex polytope graphs are created by maintaining the polytope's vertex-edge incidence relation. Imran et al. have researched the metric dimension (MD) of different types of convex polytopes [16, 17]. Slater [31] introduced the concept of the MD of a graph, followed by Harary et al. [12].

Partition dimension (PD), a generalized form of MD for graphs, is applied in areas such as computer networking, optimization [19], and chemical structure modeling [12]. Chartrand et al. developed the idea of PD of graph as an extension of MD of graph in 2000 [8]. Let $G$ be a connected graph of order $n$, where the vertex set is $V(G)$ and the edge set is $E(G)$. Let $J$ be a subset of vertices in $G$ then $d(u,w)$ represents the distance between vertices $u, w$ and $d(w,J)= \min\{d(w,y)|y \in J\}$ represents the distance between a vertex $w$ and the subset $J$.

Consider the ordered subset $\mu = \{w_1, w_2, \ldots, w_l\}$ of $V(G)$. An $l$-tuple, $r(y|\mu) = (d(y,w_1), d(y,w_2), \ldots, d(y,w_l))$ is called the representation of vertex $y$ with respect to $\mu$. The set $\mu$ is a resolving set of $G$ if the representations, $r(y|\mu)$, are distinct for each vertex of $G$.

The MD of the Cayley graph was calculated by Rezaei et al. [25]. MD for connected networks has been investigated recently in [1]. The concept of fault-tolerant metric dimension of graphs was first proposed by Hernando et al. in 2008 [13]. If $\mu \setminus \{\alpha\}$ is a resolving set for each $\alpha \in \mu$, then the resolving set $\mu$ of $V(G)$ is fault-tolerant. The fault-tolerant metric dimension of $G$ is the minimal cardinality of the fault-tolerant resolving set $\mu$ and is represented by, $\beta^{'}(G)$. Fault-tolerant metric dimension for circulant graphs has been explored by Saha et al. [27, 26].

Let $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_l\}$ be a partition with $l-$partition classes of

$V(G)$. The representation $r(w|\Theta)$ of vertex $w$ with respect to partition $\Theta$ is an $l-$vector given as $r(w|\Theta) = (d(w,\Theta_1), d(w,\Theta_2), \ldots, d(w,\Theta_l))$. If all of the vertices in $G$ have different representations, then $\Theta$ is referred to as the resolving partition of $G$. The $\mathrm{pd}(G)$ represents the PD of $G$, which is defined as $\mathrm{pd}(G) = \min\{|\Theta| : \Theta \text{ is resolving partition of } G\}$. Graphs with partition dimensions of 2 or $n$ were described by Chartrand et al. [8]. For different classes of linked networks, the PD may be infinite, bounded, or constant. For example, [14] and [18], respectively, examine the PD of the generalized Mobius ladder and a few wheel-related graphs. Asim et al. [22] examined the circulant graph $C_n(1,2,3,4)$'s PD. Chu et al. suggest sharp constraints for PD of convex polytopes [9]. The computation of MD for general graphs is NP-hard, as demonstrated by Khuller et al. [20]. The problem of PD is also NP-hard since it is a generalized version of MD.

Salman et al. established the notion of fault tolerant partition dimension (FTPD) [28]. Let $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_l\}$ be a partition with $l$ partition classes of the vertex set of connected graph $G$. This $\Theta$ is said to be fault-tolerant partition generator (FTPG) of $G$ if for each pair of distinct vertices $u, z \in V(G)$, $r(u|\Theta)$ and $r(z|\Theta)$ differ in at least two places. The FTPD of $G$, is denoted by $\mathcal{F}(G)$ and defined as, $\mathcal{F}(G) = \min\{|\Theta| : \Theta \text{ is FTPG of } G\}$. The FTPD of cyclic networks [3], tadpole and necklace graphs [4], mesh related networks [2], cycle related graphs [6], and chemical graphs [5] has been revealed by Kamran et al. They also gave applications of FTPD in scenarios of supply chain optimization problem, routing optimization problem, water flow in a locality, and sensors deployed in homes ([3, 4, 2, 6]). The use of PD and FTPD for addressing schemes for oxide interconnection networks has been discussed in [24]. Asim et al. [21] have computed the lower bound of FTPD for regular graphs. The motivation for introducing the concept of the fault-tolerant partition dimension (FTPD) stems from the growing need for resilient and robust systems in modern networked environments. In real-world applications such as sensor placement in smart cities, fault-resilient communication in distributed systems, and structural monitoring of infrastructure networks, it is crucial to maintain the integrity of network identification even in the presence of node or communication failures. FTPD ensures that even if a single partition class becomes ineffective or unavailable, the distinguishing capability of the partition remains intact. This added robustness enhances the practical utility of graph-theoretic techniques in domains such as network security, supply chain optimization, and disaster response systems. We extend this discipline in the paper by extracting the

bounds on FTPD of convex polytopes $T_n$, $R_n$, and $U_n$.

The following lemma provides a lower bound on FTPD:

**Lemma 1.1 ([23])** *A graph $G$ of order $n \geq 5$ has $\mathcal{F}(G) \geq 4$ if $G$ includes a node of degree at least 4.*

Following are some useful results on $\mathcal{F}(G)$ revealed by Salman et al.

**Proposition 1.2 ([28])** *Consider a connected graph $G$ with order $n \geq 2$,*

*(a)* $\mathrm{pd}(G) \leq \mathcal{F}(G)$.

*(b)* $\mathcal{F}(G) = n \iff G \cong K_n$ *or* $G \cong K_n - e$.

**Proposition 1.3 ([29])**

*(a) Consider a connected graph $G$ with order $n \geq 2$, $\mathcal{F}(G) \leq \beta^{'}(G) + 1$.*

*(b) For a connected graph $G$ of order $n \geq 3$, $3 \leq \mathcal{F}(G) \leq n$.*

Section 2, of the remaining part of the article is devoted for finding the bounds on $\mathcal{F}(T_n)$. In Section 3, we are concerned in finding the bounds on $\mathcal{F}(R_n)$. Section 4, includes the bounds on $\mathcal{F}(U_n)$. Application is included in Section 5. Finally, in the last section, we conclude the paper by raising an open question.

## 2 FTPD of Convex Polytope $T_n$

A convex polytope denoted by $T_n$ has 3-sided, 5-sided and $n$-sided faces. Vertices of $(T_n)$ for $n \geq 3$, are divided into three sets, $A = \{a_i : 1 \leq i \leq n\}$, $B = \{b_i : 1 \leq i \leq n\}$ and $C = \{c_i : 1 \leq i \leq n\}$. The cycle formed by $\{a_i : 1 \leq i \leq n\}$ is called inner cycle, the set of vertices $\{b_i : 1 \leq i \leq n\}$ called the central vertices and the cycle formed by $\{c_i : 1 \leq i \leq n\}$ called outer cycle. The set $V(T_n) = A \cup B \cup C$ and $E(Tn) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_1 a_n\} \cup \{a_1 b_n\} \cup \{a_i b_i : 1 \leq i \leq n\} \cup \{b_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i c_i : 1 \leq i \leq n\} \cup \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{c_1 c_n\}$ are vertex and edge sets of convex polytope $T_n$ respectively. Graph of $T_6$ is displayed in Figure 1.

The $\beta(T_n)$ has been computed in [15].

**Lemma 2.1 ([15])** *For every $n \geq 6$, $\beta(T_n) = 3$.*

The following theorem will allow us to compute the bounds on FTPD of $T_n$.

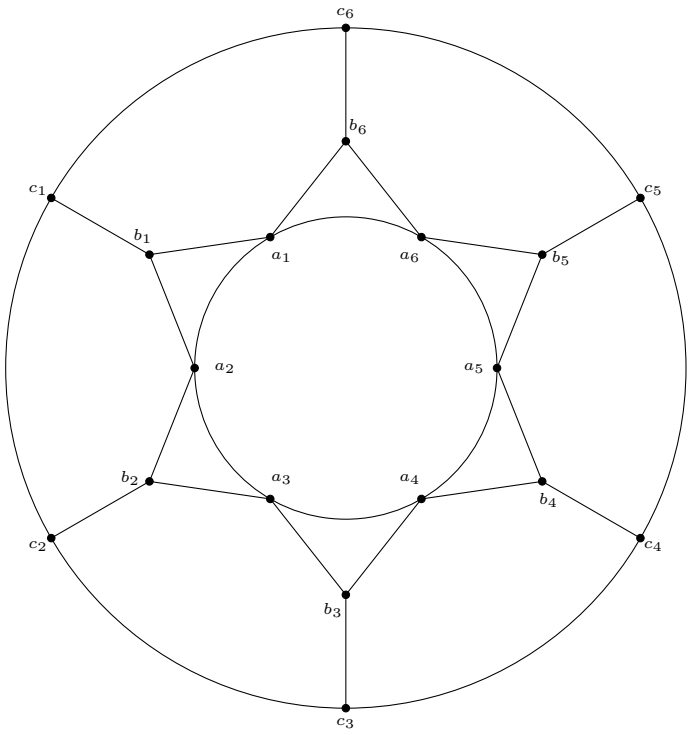


Figure 1: Convex Polytope $T_6$.

**Theorem 2.2** *Let $T_n$ be a convex polytope with $n \geq 3$, then, $4 \leq \mathcal{F}(T_n) \leq 5$.*

**Proof:** Let $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$ be a partition of $V(T_n)$ for $n = 3$, and 4. For $n = 3$, consider, $\Theta_1 = \{a_1\}, \Theta_2 = \{a_2, b_1, b_2\}, \Theta_3 = \{a_3\}$ and $\Theta_4 = \{b_3, c_1, c_2, c_3\}$, and, for $n = 4$, consider, $\Theta_1 = \{a_1, b_1\}, \Theta_2 = \{a_2, a_3, b_2\}, \Theta_3 = \{a_4, b_3, b_4\}$ and $\Theta_4 = \{c_1, c_2, c_3, c_4\}$. Easily it can be seen that $\Theta$ is a FTPG for $n = 3$ and 4.

Now suppose, $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$ be a partition of $V(T_n)$ for $n \geq 5$. For $n = 5$, consider, $\Theta_1 = \{a_i : 1 \leq i \leq 5\} \cup \{b_i : 1 \leq i \leq 5\}, \Theta_2 = \{c_1, c_2\}, \Theta_3 = \{c_3\}, \Theta_4 = \{c_4\}$ and $\Theta_5 = \{c_5\}$. For $n = 6$ and 9, consider, $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp - 1\} \cup \{c_{\wp+2}\}, \Theta_2 = \{c_\wp\}, \Theta_3 = \{c_{\wp+1}\}, \Theta_4 = \{c_i : \wp + 3 \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$. It can be verified that $\Theta$ is an FTPG for $n = 5, 6$ and 9.

**Case 1**: **When $n = 3\wp$ for $\wp \geq 4$.**

The representation of vertices of $T_n$, ($R_p(V(T_n))$ w.r.t $\Theta$ considering

$\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp - 1\} \cup \{c_{\wp+1}, c_{n-\wp-1}\}, \Theta_2 = \{c_\wp\}, \Theta_3 = \{c_i : \wp + 2 \leq i \leq n - \wp - 2\}, \Theta_4 = \{c_i : n - \wp \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$ are shown in Table 1 and representations can be generated by substituting appropriate values.

Table 1: $R_p(V(T_n))$ **w.r.t** $\Theta$ **considering for** $n = 3\wp, \wp \geq 4$

| **Distance of vertex** | **from** $\Theta_1$ | **from** $\Theta_2$ | **from** $\Theta_3$ | **from** $\Theta_4$ | **from** $\Theta_5$ |
|---|---|---|---|---|---|
| $a_i(1 \leq i \leq \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 4 - i$ | $i + 2$ | $i + 1$ |
| $a_{\wp+1}$ | 0 | 2 | 3 | $\wp + 1$ | $\wp + 2$ |
| $a_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $i + 2$ | 2 | $\wp + 1 - i$ | $\wp + 2 + i$ |
| $a_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 2 - i$ | $\lfloor \frac{n}{2} \rfloor + 2 - i$ |
| $a_{2\wp-1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\wp + i$ | $i + 2$ | 2 | $\wp + 3 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lceil \frac{\wp}{2} \rceil + i + 3$ | 2 | $\lfloor \frac{\wp}{2} \rfloor - i + 2$ |
| $b_i(1 \leq i \leq \wp - 1)$ | 0 | $\wp + 1 - i$ | $\wp + 3 - i$ | $i + 2$ | $i + 1$ |
| $b_\wp$ | 0 | 1 | 3 | $\wp + 1$ | $\wp + 1$ |
| $b_{\wp+1}$ | 0 | 2 | 2 | $\wp$ | $\wp + 2$ |
| $b_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $i + 2$ | 2 | $\wp - i$ | $\wp + 2 + i$ |
| $b_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 2)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 1$ | 1 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ | $\lceil \frac{n}{2} \rceil + 1 - i$ |
| $b_{2\wp-1}$ | 0 | $\wp$ | 2 | 2 | $\wp + 2$ |
| $b_{2\wp-1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\wp + i$ | $i + 2$ | 1 | $\wp - i + 2$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i + 3$ | 1 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $b_n$ | 0 | $\wp + 1$ | $\wp + 3$ | 2 | 1 |
| $c_i(1 \leq i \leq \wp - 1)$ | 0 | $\wp - i$ | $\wp + 2 - i$ | $i + 1$ | $i$ |
| $c_\wp$ | 1 | 0 | 2 | $\wp$ | $\wp$ |
| $c_{\wp+1}$ | 0 | 1 | 1 | $\wp - 1$ | $\wp + 1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor - 1)$ | 1 | $i + 1$ | 0 | $\wp - i - 1$ | $\wp + 1 + i$ |
| $c_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 2)$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + i$ | 0 | $\lceil \frac{\wp}{2} \rceil - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $c_{2\wp-1}$ | 0 | $\wp - 1$ | 1 | 1 | $\wp + 1$ |
| $c_{2\wp-1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ | 1 | $\wp + i - 1$ | $i + 1$ | 0 | $\wp - i + 1$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\lceil \frac{n}{2} \rceil - i$ | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 0 | $\lceil \frac{\wp}{2} \rceil - i$ |
| $c_n$ | 1 | $\wp$ | $\wp + 2$ | 1 | 0 |

**Case 2**: **When** $n = 3\wp + 1$ **for** $\wp \geq 2$**.**

The $(R_p(V(T_n))$ w.r.t $\Theta$ considering $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp - 1\} \cup \{c_{\wp+1}\}, \Theta_2 = \{c_\wp\}, \Theta_3 = \{c_i : \wp + 2 \leq i \leq n - \wp - 1\}, \Theta_4 = \{c_i : n - \wp \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$ are shown in Table 2 which can be verified by substituting appropriate values.

Table 2: $R_p(V(T_n))$ for $n = 3\wp + 1, \wp \geq 2$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_i(1 \leq i \leq \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 4 - i$ | $i + 2$ | $i + 1$ |
| $a_{\wp+1}$ | 0 | 2 | 3 | $\wp + 2$ | $\wp + 2$ |
| $a_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $i + 2$ | 2 | $\wp + 2 - i$ | $\wp + 2 + i$ |
| $a_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 2 | $\lceil \frac{\wp}{2} \rceil + 2 - i$ | $\lfloor \frac{n}{2} \rfloor + 2 - i$ |
| $a_{2\wp+i+1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\wp + i + 2$ | $i + 2$ | 2 | $\wp + 2 - i$ |
| $a_{n-\lceil \frac{\wp}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 2 | $\lceil \frac{\wp}{2} \rceil - i + 2$ |
| $b_i(1 \leq i \leq \wp)$ | 0 | $\wp + 1 - i$ | $\wp + 3 - i$ | $i + 2$ | $i + 1$ |
| $b_{\wp+1}$ | 0 | 2 | 2 | $\wp + 1$ | $\wp + 2$ |
| $b_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $i + 2$ | 1 | $\wp - i + 1$ | $\wp + 2 + i$ |
| $b_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 1 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ | $\lfloor \frac{n}{2} \rfloor + 1 - i$ |
| $b_{2\wp+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\wp + i + 1$ | $i + 1$ | 1 | $\wp - i + 2$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor + i - 1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lceil \frac{n}{2} \rceil - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 1 | $\lfloor \frac{\wp}{2} \rfloor - i + 2$ |
| $b_n$ | 0 | $\wp + 1$ | $\wp + 2$ | 2 | 1 |
| $c_i(1 \leq i \leq \wp - 1)$ | 0 | $\wp - i$ | $\wp + 2 - i$ | $i + 1$ | $i$ |
| $c_\wp$ | 1 | 0 | 2 | $\wp + 1$ | $\wp$ |
| $c_{\wp+1}$ | 0 | 1 | 1 | $\wp$ | $\wp + 1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 1 | $i + 1$ | 0 | $\wp - i$ | $\wp + 1 + i$ |
| $c_{\lfloor \frac{n}{2} \rfloor + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lceil \frac{\wp}{2} \rceil + i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + 1 - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $c_{2\wp+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\wp + i$ | $i$ | 0 | $\wp - i + 1$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor + i - 1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lceil \frac{n}{2} \rceil - i$ | $\lceil \frac{\wp}{2} \rceil + i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $c_n$ | 1 | $\wp$ | $\wp + 1$ | 1 | 0 |

**Case 3**: **When $n = 3\wp + 2$ for $\wp \geq 2$.**

The $(R_p(V(T_n))$ w.r.t $\Theta$ considering $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp\}, \Theta_2 = \{c_{\wp+1}\}, \Theta_3 = \{c_i : \wp + 2 \leq i \leq n - \wp - 1\}, \Theta_4 = \{c_i : n - \wp \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$ are shown in Table 3 which can be verified by substituting appropriate values.

Table 3: $R_p(V(T_n))$ for $n = 3\wp + 2, \wp \geq 2$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_1$ | 0 | $\wp + 2$ | $\wp + 3$ | 3 | 2 |
| $a_{i+1}(1 \leq i \leq \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 3 - i$ | $i + 3$ | $i + 2$ |
| $a_{\wp+2}$ | 0 | 2 | 3 | $\wp + 2$ | $\wp + 2$ |
| $a_{\wp+2+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $i + 2$ | 2 | $\wp + 2 - i$ | $\wp + 3 + i$ |
| $a_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 3 - i$ | $\lfloor \frac{n}{2} \rfloor + 2 - i$ |
| $a_{2\wp+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\wp + i + 1$ | $i + 1$ | 2 | $\wp + 3 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lceil \frac{\wp}{2} \rceil + i + 2$ | 2 | $\lfloor \frac{\wp}{2} \rfloor - i + 2$ |

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $b_i(1 \leq i \leq \wp)$ | 0 | $\wp+2-i$ | $\wp+3-i$ | $i+2$ | $i+1$ |
| $b_{\wp+1}$ | 0 | 1 | 2 | $\wp+2$ | $\wp+2$ |
| $b_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $i+1$ | 1 | $\wp-i+2$ | $\wp+2+i$ |
| $b_{\lfloor \frac{n}{2} \rfloor+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor+i+1$ | 1 | $\lceil \frac{\wp}{2} \rceil+2-i$ | $\lceil \frac{n}{2} \rceil+1-i$ |
| $b_{2\wp+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\wp+i+1$ | $i+1$ | 1 | $\wp-i+2$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor-i+2$ | $\lceil \frac{\wp}{2} \rceil+i+1$ | 1 | $\lfloor \frac{\wp}{2} \rfloor-i+2$ |
| $b_n$ | 0 | $\wp+2$ | $\wp+2$ | 2 | 1 |
| $c_i(1 \leq i \leq \wp)$ | 0 | $\wp-i+1$ | $\wp+2-i$ | $i+1$ | $i$ |
| $c_{\wp+1}$ | 1 | 0 | 1 | $\wp+1$ | $\wp+1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $i$ | 0 | $\wp-i+1$ | $\wp+1+i$ |
| $c_{\lfloor \frac{n}{2} \rfloor+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\lfloor \frac{\wp}{2} \rfloor+i$ | 0 | $\lceil \frac{\wp}{2} \rceil+1-i$ | $\lceil \frac{n}{2} \rceil-i$ |
| $c_{2\wp+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\wp+i$ | $i$ | 0 | $\wp-i+1$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lfloor \frac{n}{2} \rfloor-i+1$ | $\lceil \frac{\wp}{2} \rceil+i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor-i+1$ |
| $c_n$ | 1 | $\wp+1$ | $\wp+1$ | 1 | 0 |

Tables 1-3, justifies that $\Theta$ is FTPG of $T_n$, thus by Lemma 1.1 we have, $4 \leq \mathcal{F}(T_n) \leq 5$. □

**Example 2.3** *To verify Table 2, consider a convex polytope $T_7$, with partition set $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$, where $\Theta_1 = \{a_i : 1 \leq i \leq 7\} \cup \{b_i : 1 \leq i \leq 7\} \cup \{c_1, c_3\}, \Theta_2 = \{c_2\}, \Theta_3 = \{c_4\}, \Theta_4 = \{c_5, c_6\}$ and $\Theta_5 = \{c_7\}$.*

*Representation of vertices of $T_7$, are as follows:*

$r(a_1|\Theta) = (0,3,5,3,2)$, $r(a_2|\Theta) = (0,2,4,4,3)$, $r(a_3|\Theta) = (0,2,3,4,4)$, $r(a_4|\Theta) = (0,3,2,3,5)$, $r(a_5|\Theta) = (0,4,2,2,4)$, $r(a_6|\Theta) = (0,5,3,2,3)$, $r(a_7|\Theta) = (0,4,4,2,2)$.

$r(b_1|\Theta) = (0,2,4,3,2)$, $r(b_2|\Theta) = (0,1,3,4,3)$, $r(b_3|\Theta) = (0,2,2,3,4)$, $r(b_4|\Theta) = (0,3,1,2,5)$, $r(b_5|\Theta) = (0,4,2,1,3)$, $r(b_6|\Theta) = (0,4,3,1,2)$, $r(b_7|\Theta) = (0,3,4,2,1)$.

$r(c_1|\Theta) = (0,1,3,2,1)$, $r(c_2|\Theta) = (1,0,2,3,2)$, $r(c_3|\Theta) = (0,1,1,2,3)$, $r(c_4|\Theta) = (1,2,0,1,3)$, $r(c_5|\Theta) = (1,3,1,0,2)$, $r(c_6|\Theta) = (1,3,2,0,1)$, $r(c_7|\Theta) = (1,2,3,1,0)$.

*It is clear from above distinct representations that $\Theta$ is fault tolerant partition generator of $T_7$. Remaining 8 tables can be verified for different values of $n$.*

## 3 FTPD of Convex Polytope $R_n$

A convex polytope denoted by $R_n$ has $4n$ 3-sided faces. Vertices of $(T_n)$ for $n \geq 3$, are divided into three sets, $A = \{a_i : 1 \leq i \leq n\}$, $B = \{b_i : 1 \leq i \leq n\}$ and $C = \{c_i : 1 \leq i \leq n\}$. The cycle formed by $\{a_i : 1 \leq i \leq n\}$ is called inner

cycle, the cycle formed by $\{b_i : 1 \leq i \leq n\}$ is the central cycle and the cycle formed by $\{c_i : 1 \leq i \leq n\}$ is called outer cycle. The set $V(T_n) = A \cup B \cup C$ and $E(Tn) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_1 a_n\} \cup \{a_1 b_n\} \cup \{a_i b_i 1 \leq i \leq n\} \cup \{b_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i c_i : 1 \leq i \leq n\} \cup \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{c_1 c_n\}$ are vertex and edge sets of convex polytope $R_n$ respectively. Graph of $R_6$ is displayed in Figure 2.

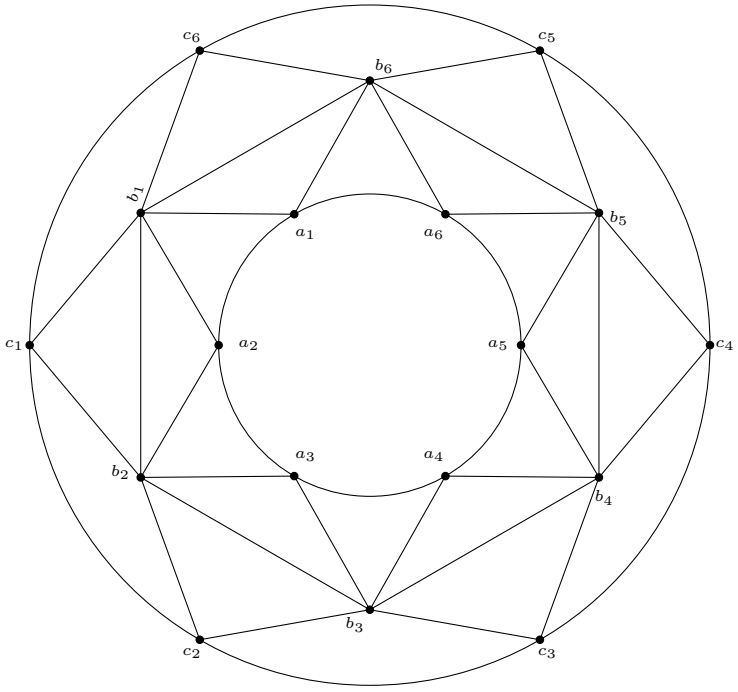


Figure 2: Convex Polytope $R_6$.

The following theorem will allow us to compute the bounds on FTPD of $R_n$.

**Theorem 3.1** *Let $R_n$ be a convex polytope with $n \geq 3$, then, $4 \leq \mathcal{F}(R_n) \leq 5$.*

**Proof:** Let $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$ be a partition of $V(R_n)$ for $n = 3$ and 4. For $n = 3$, consider, $\Theta_1 = \{a_1, a_2, a_3, b_1\}, \Theta_2 = \{b_2, c_2\}, \Theta_3 = \{b_3, c_3\}$ and $\Theta_4 = \{c_1\}$. For $n = 4$, consider, $\Theta_1 = \{a_1, a_2, a_3, b_1\}, \Theta_2 = \{a_4, b_4, c_3, c_4\}, \Theta_3 = \{b_2, b_3, c_2\}$ and $\Theta_4 = \{c_1\}$. Easily it can be verified that $\Theta$ is a FTPG for $n = 3$ and 4.

Now suppose, $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$ be a partition of $V(R_n)$ for $n \geq 5$. For $n = 5$, consider, $\Theta_1 = \{a_i : 1 \leq i \leq 5\} \cup \{b_i : 1 \leq i \leq 4\}, \Theta_2 =$

$\{b_5, c_4, c_5\}, \Theta_3 = \{c_1\}, \Theta_4 = \{c_2\}$ and $\Theta_5 = \{c_3\}$.

For $n = 6$, consider, $\Theta_1 = \{a_i : 1 \leq i \leq 6\} \cup \{b_i : 1 \leq i \leq 6\} \cup \{c_1, c_4\}, \Theta_2 = \{c_2\}, \Theta_3 = \{c_3\}, \Theta_4 = \{c_5\}$ and $\Theta_5 = \{f_6\}$.

For $n = 9$, consider, $\Theta_1 = \{a_i : 1 \leq i \leq 9\} \cup \{b_i : 1 \leq i \leq 9\} \cup \{c_1, c_2\}, \Theta_2 = \{c_3\}, \Theta_3 = \{c_4, c_5\}, \Theta_4 = \{c_6, c_7, c_8\}$ and $\Theta_5 = \{c_9\}$.

It can be confirmed that $\Theta$ is FTPG for $n = 5, 6$ and 9.

**Case 1**: **When $n = 3\wp$ for $\wp \geq 4$.**

The $r(v|\Theta)$ with respect to $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp - 1\} \cup \{c_{\wp+1}, c_{n-\wp-1}\}, \Theta_2 = \{c_\wp\}, \Theta_3 = \{c_i : \wp + 2 \leq i \leq n - \wp - 2\}, \Theta_4 = \{c_i : n - \wp \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$ are shown in Table 4 which can be verified by substituting appropriate values.

Table 4: $R_p(V(R_n))$ for $n = 3\wp, \wp \geq 4$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_1$ | 0 | $\wp+1$ | $\wp+3$ | 2 | 2 |
| $a_{i+1}(1 \leq i \leq \wp-1)$ | 0 | $\wp+1-i$ | $\wp+3-i$ | $i+2$ | $i+1$ |
| $a_{\wp+1}$ | 0 | 2 | 3 | $\wp+1$ | $\wp+1$ |
| $a_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $i+1$ | 2 | $\wp+1-i$ | $\wp+1+i$ |
| $a_{\lfloor \frac{n}{2} \rfloor+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 1$ | 2 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ | $\lceil \frac{n}{2} \rceil + 1 - i$ |
| $a_{2\wp+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $\wp+i$ | $i+2$ | 2 | $\wp+2-i$ |
| $a_{n-\lceil \frac{\wp}{2} \rceil+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\lceil \frac{n}{2} \rceil - i + 1$ | $\lfloor \frac{\wp}{2} \rfloor + i + 3$ | 2 | $\lceil \frac{\wp}{2} \rceil - i + 1$ |
| $b_i(1 \leq i \leq \wp)$ | 0 | $\wp+1-i$ | $\wp+3-i$ | $i+1$ | $i$ |
| $b_{\wp+1}$ | 0 | 1 | 2 | $\wp$ | $\wp+1$ |
| $b_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $i+1$ | 1 | $\wp-i$ | $\wp+1+i$ |
| $b_{\lceil \frac{n}{2} \rceil+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + 1 - i$ | $\lfloor \frac{n}{2} \rfloor + 1 - i$ |
| $b_{2\wp-1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\wp+i-1$ | $i+1$ | 1 | $\wp-i+2$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i + 2$ | 1 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $c_i(1 \leq i \leq \wp-1)$ | 0 | $\wp-i$ | $\wp+2-i$ | $i+1$ | $i$ |
| $c_\wp$ | 1 | 0 | 2 | $\wp$ | $\wp$ |
| $c_{\wp+1}$ | 0 | 1 | 1 | $\wp-1$ | $\wp+1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor - 1)$ | 1 | $i+1$ | 0 | $\wp-i-1$ | $\wp+1+i$ |
| $c_{\lceil \frac{n}{2} \rceil+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 2)$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + i$ | 0 | $\lceil \frac{\wp}{2} \rceil - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $c_{2\wp-1}$ | 0 | $\wp-1$ | 1 | 1 | $\wp+1$ |
| $c_{2\wp-1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ | 1 | $\wp+i-1$ | $i+1$ | 0 | $\wp-i+1$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\lceil \frac{n}{2} \rceil - i$ | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 0 | $\lceil \frac{\wp}{2} \rceil - i$ |
| $c_n$ | 1 | $\wp$ | $\wp+2$ | 1 | 0 |

**Case 2**: **When $n = 3\wp + 1$ for $\wp \geq 2$.**

The $r(v|\Theta)$ with respect to $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp - 1\} \cup \{c_{\wp+1}\}, \Theta_2 = \{c_\wp\}, \Theta_3 = \{c_i : \wp + 2 \leq i \leq n - \wp - 1\}, \Theta_4 = \{c_i : n - \wp \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$ are shown in Table 5.

Table 5: $R_p(V(R_n))$ for $n = 3\wp + 1, \wp \geq 2$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_1$ | 0 | $\wp + 1$ | $\wp + 2$ | 2 | 2 |
| $a_{i+1}(1 \leq i \leq \wp - 1)$ | 0 | $\wp + 1 - i$ | $\wp + 3 - i$ | $i + 2$ | $i + 1$ |
| $a_{\wp+1}$ | 0 | 2 | 3 | $\wp + 2$ | $\wp + 1$ |
| $a_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $i + 1$ | 2 | $\wp + 2 - i$ | $\wp + 1 + i$ |
| $a_{\lfloor \frac{n}{2} \rfloor + i + 1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 2 - i$ | $\lceil \frac{n}{2} \rceil + 1 - i$ |
| $a_{2\wp+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\wp + i + 1$ | $i + 1$ | 2 | $\wp + 2 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lceil \frac{n}{2} \rceil - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 2 | $\lfloor \frac{\wp}{2} \rfloor - i + 2$ |
| $b_i(1 \leq i \leq \wp)$ | 0 | $\wp + 1 - i$ | $\wp + 3 - i$ | $i + 1$ | $i$ |
| $b_{\wp+1}$ | 0 | 1 | 2 | $\wp + 1$ | $\wp + 1$ |
| $b_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $i + 1$ | 1 | $\wp - i + 1$ | $\wp + 1 + r$ |
| $b_{\lceil \frac{n}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 1$ | 1 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ | $\lfloor \frac{n}{2} \rfloor + 1 - i$ |
| $b_{2\wp+i+1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\wp + i + 1$ | $i + 1$ | 1 | $\wp - i + 1$ |
| $b_{n-\lceil \frac{\wp}{2} \rceil + i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lfloor \frac{\wp}{2} \rfloor + i + 1$ | 1 | $\lceil \frac{\wp}{2} \rceil - i + 1$ |
| $c_i(1 \leq i \leq \wp - 1)$ | 0 | $\wp - i$ | $\wp + 2 - i$ | $i + 1$ | $i$ |
| $c_\wp$ | 1 | 0 | 2 | $\wp + 1$ | $\wp$ |
| $c_{\wp+1}$ | 0 | 1 | 1 | $\wp$ | $\wp + 1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 1 | $i + 1$ | 0 | $\wp - i$ | $\wp + 1 + i$ |
| $c_{\lfloor \frac{n}{2} \rfloor + i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lceil \frac{\wp}{2} \rceil + i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + 1 - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $c_{2\wp+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\wp + i$ | $i$ | 0 | $\wp - i + 1$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor + i - 1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lceil \frac{n}{2} \rceil - i$ | $\lceil \frac{\wp}{2} \rceil + i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $c_n$ | 1 | $\wp$ | $\wp + 1$ | 1 | 0 |

**Case 3**: **When $n = 3\wp+2$ for $\wp \geq 2$.** The $r(v|\Theta)$ with respect to $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq \wp\}, \Theta_2 = \{c_{\wp+1}\}, \Theta_3 = \{c_i : \wp + 2 \leq i \leq n - \wp - 1\}, \Theta_4 = \{c_i : n - \wp \leq i \leq n - 1\}$ and $\Theta_5 = \{c_n\}$ are shown in Table 6 which can be verified by substituting appropriate values.

Table 6: $R_p(V(R_n))$ for $n = 3\wp + 2, \wp \geq 2$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_1$ | 0 | $\wp + 2$ | $\wp + 2$ | 2 | 2 |
| $a_{i+1}(1 \leq i \leq \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 3 - i$ | $i + 2$ | $i + 1$ |
| $a_{\wp+2}$ | 0 | 2 | 2 | $\wp + 2$ | $\wp + 2$ |
| $a_{\wp+2+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $i + 1$ | 2 | $\wp + 2 - i$ | $\wp + 2 + i$ |

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_{\lfloor \frac{n}{2} \rfloor+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 1$ | 2 | $\lceil \frac{\wp}{2} \rceil + 2 - i$ | $\lceil \frac{n}{2} \rceil + 1 - i$ |
| $a_{2\wp+i+2}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $\wp + i + 1$ | $i + 1$ | 2 | $\wp + 2 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\lceil \frac{n}{2} \rceil - i + 1$ | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 2 | $\lceil \frac{\wp}{2} \rceil - i + 1$ |
| $b_i(1 \leq i \leq \wp + 1)$ | 0 | $\wp + 2 - i$ | $\wp + 3 - i$ | $i + 1$ | $i$ |
| $b_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $i$ | 1 | $\wp - i + 2$ | $\wp + 1 + i$ |
| $b_{\lfloor \frac{n}{2} \rfloor+i+1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + 2 - i$ | $\lfloor \frac{n}{2} \rfloor + 1 - i$ |
| $b_{2\wp+i+2}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\wp + i + 1$ | $i + 1$ | 1 | $\wp - i + 2$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 1 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $c_i(1 \leq i \leq \wp)$ | 0 | $\wp - i + 1$ | $\wp + 2 - i$ | $i + 1$ | $i$ |
| $c_{\wp+1}$ | 1 | 0 | 1 | $\wp + 1$ | $\wp + 1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $i$ | 0 | $\wp - i + 1$ | $\wp + 1 + i$ |
| $c_{\lfloor \frac{n}{2} \rfloor+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + i$ | 0 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $c_{2\wp+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\wp + i$ | $i$ | 0 | $\wp - i + 1$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $c_n$ | 1 | $\wp + 1$ | $\wp + 1$ | 1 | 0 |

Tables 4-6, justifies that $\Theta$ is FTPG of $R_n$, thus by Lemma 1.1 we have, $4 \leq \mathcal{F}(R_n) \leq 5$. □

# 4 FTPD of Convex Polytope $U_n$

A convex polytope denoted by $U_n$ has 3-sided, 4-sided and 5-sided faces. Vertices of $U_n$ for $n \geq 3$, are divided into four sets, $A = \{a_i : 1 \leq i \leq n\}$, $B = \{b_i : 1 \leq i \leq n\}$, $C = \{c_i : 1 \leq i \leq n\}$ and $D = \{d_i : 1 \leq i \leq n\}$. The cycles formed by $\{a_i : 1 \leq i \leq n\}$, $\{b_i : 1 \leq i \leq n\}$, and $\{d_i : 1 \leq i \leq n\}$ are called the interior cycle, the center cycle, and the exterior cycle respectively. The set of vertices $\{c_i : 1 \leq i \leq n\}$ is called the set of interior vertices. The set $V(U_n) = A \cup B \cup C \cup D$ and $E(Un) = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_1 a_n\} \cup \{a_i b_i : 1 \leq i \leq n\} \cup \{b_i b_{i+1} : 1 \leq i \leq n-1\} \cup \{b_1 b_n\} \cup \{b_i c_i : 1 \leq i \leq n\} \cup \{c_i d_i : 1 \leq i \leq n\} \cup \{c_{i+1} d_i : 1 \leq i \leq n-1\} \cup \{c_1 d_n\} \cup \{d_i d_{i+1} : 1 \leq i \leq n-1\} \cup \{d_1 d_n\}$ are vertex and edge sets of convex polytope $U_n$ respectively. Graph of $U_6$ is displayed in Figure 3.

The following theorem will allow us to compute the bounds on FTPD of $U_n$.

**Theorem 4.1** *Let $U_n$ be a convex polytope with $n \geq 3$, then, $4 \leq \mathcal{F}(U_n) \leq 5$.*

**Proof:** Let $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$ be a partition of $V(U_n)$ for $n = 3$ and 4.

When $n = 3$, consider, $\Theta_1 = \{a_i : 1 \leq i \leq 3\} \cup \{b_i : 1 \leq i \leq 3\} \cup \{c_i : 1 \leq i \leq 3\}, \Theta_2 = \{d_1\}, \Theta_3 = \{d_2\}$ and $\Theta_4 = \{d_3\}$.

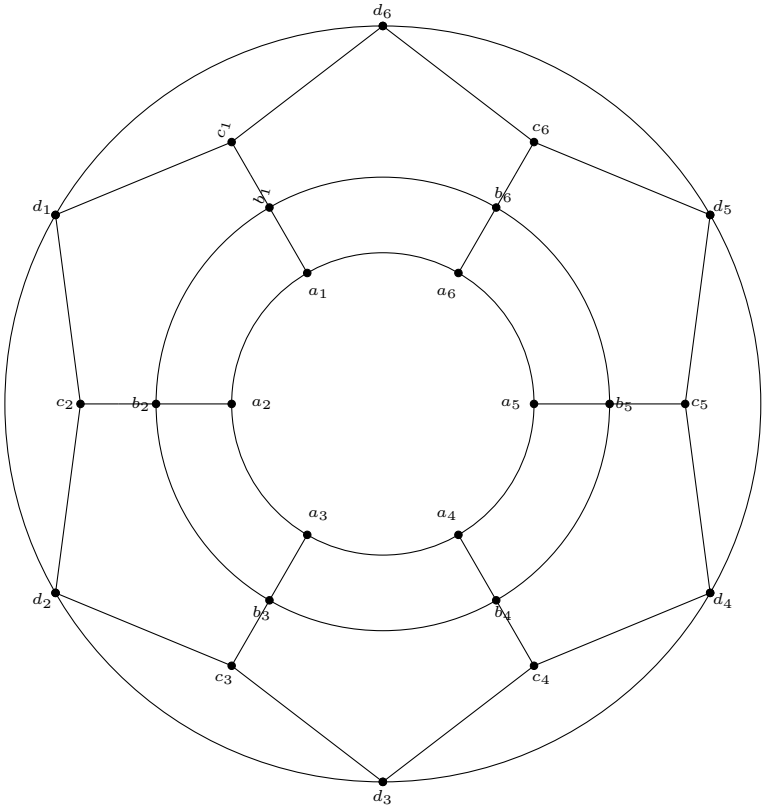


Figure 3: Convex Polytope $U_6$.

When $n = 4$, consider, $\Theta_1 = \{a_i : 1 \leq i \leq 4\} \cup \{b_1, b_2\}, \Theta_2 = \{b_3, b_4, c_3, c_4, d_3\}, \Theta_3 = \{c_1, d_1, d_4\}$ and $\Theta_4 = \{c_2, d_2\}$.

It is obvious that $\Theta$ is FTPG for $n = 3$ and 4.

Now suppose, $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$ be the partition set of vertices of $U_n$. The $r(v|\Theta)$ for $n \geq 5$, considering $\Theta_1 = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq \wp - 1\}, \Theta_2 = \{d_\wp\}, \Theta_3 = \{d_i : \wp + 1 \leq i \leq n - \wp\}, \Theta_4 = \{d_i : n - \wp + 1 \leq i \leq n - 1\}$ and $\Theta_5 = \{d_n\}$ are given in Tables 7-9 which can be verified by substituting appropriate values.

Table 7: $R_p(V(U_n))$ for $n = 3\wp, \wp \geq 2$

| **Distance of vertex** | **from** $\Theta_1$ | **from** $\Theta_2$ | **from** $\Theta_3$ | **from** $\Theta_4$ | **from** $\Theta_5$ |
|---|---|---|---|---|---|
| $a_i (1 \leq i \leq \wp)$ | 0 | $\wp + 3 - i$ | $\wp + 4 - i$ | $i + 3$ | $i + 2$ |

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_{\wp+1}(1 \le i \le \lceil \frac{\wp}{2} \rceil)$ | 0 | $i+2$ | 3 | $\wp - i + 4$ | $\wp + 2 + i$ |
| $a_{\lceil \frac{n}{2} \rceil + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 2$ | 3 | $\lfloor \frac{\wp}{2} \rfloor + 4 - i$ | $\lfloor \frac{n}{2} \rfloor + 3 - i$ |
| $a_{2\wp+1+i}(1 \le i \le \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\wp + i + 3$ | $i+3$ | 3 | $\wp + 2 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 3$ | $\lceil \frac{\wp}{2} \rceil + 2 + i$ | 3 | $\lfloor \frac{\wp}{2} \rfloor + 3 - i$ |
| $a_{2\wp-1+i}(1 \le i \le \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\wp + i$ | $i+2$ | 2 | $\wp + 3 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lceil \frac{\wp}{2} \rceil + i + 3$ | 2 | $\lfloor \frac{\wp}{2} \rfloor - i + 2$ |
| $b_i(1 \le i \le \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 3 - i$ | $i+2$ | $i+1$ |
| $b_{\wp+1}(1 \le i \le \lceil \frac{\wp}{2} \rceil)$ | 0 | $i+1$ | 2 | $\wp - i + 3$ | $\wp + 1 + i$ |
| $b_{\lceil \frac{n}{2} \rceil + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 1$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 3 - i$ | $\lfloor \frac{n}{2} \rfloor + 2 - i$ |
| $b_{2\wp+1+i}(1 \le i \le \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\wp + i + 2$ | $i+2$ | 2 | $\wp + 1 - i$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lceil \frac{\wp}{2} \rceil + 1 + i$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 2 - i$ |
| $c_i(1 \le i \le \wp)$ | 0 | $\wp + 1 - i$ | $\wp + 2 - i$ | $i+1$ | $i$ |
| $c_{\wp+1}(1 \le i \le \lceil \frac{\wp}{2} \rceil)$ | 0 | $i$ | 1 | $\wp - i + 2$ | $\wp + i$ |
| $c_{\lceil \frac{n}{2} \rceil + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $\lceil \frac{\wp}{2} \rceil + i$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + 2 - i$ | $\lfloor \frac{n}{2} \rfloor + 1 - i$ |
| $c_{2\wp+1+i}(1 \le i \le \lceil \frac{\wp}{2} \rceil - 1)$ | 0 | $\wp + i + 1$ | $i+1$ | 1 | $\wp - i$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lceil \frac{\wp}{2} \rceil + i$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + 1 - i$ |
| $d_i(1 \le i \le \wp - 1)$ | 0 | $\wp - i$ | $\wp + 1 - i$ | $i+1$ | $i$ |
| $d_\wp$ | 1 | 0 | 1 | $\wp + 1$ | $\wp$ |
| $d_{\wp+1}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $i$ | 0 | $\wp - i + 1$ | $\wp + i$ |
| $d_{\lfloor \frac{n}{2} \rfloor + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil)$ | 1 | $\lfloor \frac{\wp}{2} \rfloor + i$ | 0 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $d_{2\wp+i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\wp + i$ | $i$ | 0 | $\wp - i$ |
| $d_{n-\lceil \frac{\wp}{2} \rceil + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil - 1)$ | 1 | $\lceil \frac{n}{2} \rceil - i$ | $\lfloor \frac{\wp}{2} \rfloor + i$ | 0 | $\lceil \frac{\wp}{2} \rceil - i$ |
| $d_n$ | 1 | $\wp$ | $\wp$ | 1 | 0 |

Table 8: $R_p(V(U_n))$ for $n = 3\wp + 1, \wp \ge 2$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_i(1 \le i \le \wp)$ | 0 | $\wp + 3 - i$ | $\wp + 4 - i$ | $i+3$ | $i+2$ |
| $a_{\wp+1}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $i+2$ | 3 | $\wp - i + 5$ | $\wp + 2 + i$ |
| $a_{\lceil \frac{n}{2} \rceil + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 3$ | 3 | $\lceil \frac{\wp}{2} \rceil + 4 - i$ | $\lfloor \frac{n}{2} \rfloor + 3 - i$ |
| $a_{2\wp+2+i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $\wp + i + 4$ | $i+3$ | 3 | $\wp + 2 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 3$ | $\lfloor \frac{\wp}{2} \rfloor + 2 + i$ | 3 | $\lfloor \frac{\wp}{2} \rfloor + 3 - i$ |
| $b_i(1 \le i \le \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 3 - i$ | $i+2$ | $i+1$ |
| $b_{\wp+1}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $i+1$ | 2 | $\wp - i + 3$ | $\wp + 1 + i$ |
| $b_{\lceil \frac{n}{2} \rceil + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 2$ | 2 | $\lceil \frac{\wp}{2} \rceil + 3 - i$ | $\lfloor \frac{n}{2} \rfloor + 2 - i$ |
| $b_{2\wp+2+i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $\wp + i + 3$ | $i+2$ | 2 | $\wp + 1 - i$ |
| $b_{n-\lceil \frac{\wp}{2} \rceil + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lfloor \frac{\wp}{2} \rfloor + 1 + i$ | 2 | $\lceil \frac{\wp}{2} \rceil + 2 - i$ |
| $c_i(1 \le i \le \wp)$ | 0 | $\wp + 1 - i$ | $\wp + 2 - i$ | $i+1$ | $i$ |
| $c_{\wp+1}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor + 1)$ | 0 | $i$ | 1 | $\wp - i + 3$ | $\wp + i$ |
| $c_{\lceil \frac{n}{2} \rceil + i}(1 \le i \le \lceil \frac{\wp}{2} \rceil + 1)$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + i + 1$ | 1 | $\lceil \frac{\wp}{2} \rceil + 2 - i$ | $\lfloor \frac{n}{2} \rfloor + 1 - i$ |
| $c_{2\wp+2+i}(1 \le i \le \lfloor \frac{\wp}{2} \rfloor - 1)$ | 0 | $\wp + i + 2$ | $i+1$ | 1 | $\wp - i$ |

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $c_{n-\lceil \frac{\wp}{2} \rceil+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 1$ | $\lfloor \frac{\wp}{2} \rfloor + i$ | 1 | $\lceil \frac{\wp}{2} \rceil + 1 - i$ |
| $d_i(1 \leq i \leq \wp - 1)$ | 0 | $\wp - i$ | $\wp + 1 - i$ | $i + 1$ | $i$ |
| $d_\wp$ | 1 | 0 | 1 | $\wp + 1$ | $\wp$ |
| $d_{\wp+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $i$ | 0 | $\wp - i + 2$ | $\wp + i$ |
| $d_{\lfloor \frac{n}{2} \rfloor+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ | 1 | $\lceil \frac{\wp}{2} \rceil + i$ | 0 | $\lfloor \frac{\wp}{2} \rfloor + 2 - i$ | $\lceil \frac{n}{2} \rceil - i$ |
| $d_{2\wp+i+1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 1)$ | 1 | $\wp + i + 1$ | $i$ | 0 | $\wp - i$ |
| $d_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor)$ | 1 | $\lceil \frac{n}{2} \rceil - i$ | $\lceil \frac{\wp}{2} \rceil + i - 1$ | 0 | $\lfloor \frac{\wp}{2} \rfloor - i + 1$ |
| $d_n$ | 1 | $\wp$ | $\wp$ | 1 | 0 |

Table 9: $R_p(V(U_n))$ for $n = 3\wp + 2, \wp \geq 1$

| **Distance of vertex** | **from $\Theta_1$** | **from $\Theta_2$** | **from $\Theta_3$** | **from $\Theta_4$** | **from $\Theta_5$** |
|---|---|---|---|---|---|
| $a_i(1 \leq i \leq \wp)$ | 0 | $\wp + 3 - i$ | $\wp + 4 - i$ | $i + 3$ | $i + 2$ |
| $a_{\wp+1}$ | 0 | 3 | 3 | $\wp + 4$ | $\wp + 3$ |
| $a_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ for $\wp \geq 2$ | 0 | $i + 3$ | 3 | $\wp - i + 5$ | $\wp + 3 + i$ |
| $a_3$ for $\wp = 1$ | 0 | 4 | 3 | 4 | 5 |
| $a_4$ for $\wp = 1$ | 0 | 5 | 3 | 3 | 4 |
| $a_5$ for $\wp = 1$ | 0 | 4 | 4 | 3 | 3 |
| $a_5$ for $\wp = 2$ | 0 | 5 | 3 | 5 | 6 |
| $a_6$ for $\wp = 2$ | 0 | 6 | 3 | 4 | 5 |
| $a_{\lceil \frac{n}{2} \rceil+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 2$ for $\wp \geq 3$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 3$ | 3 | $\lfloor \frac{\wp}{2} \rfloor + 5 - i$ | $\lfloor \frac{n}{2} \rfloor + 3 - i$ |
| $a_{2\wp+3+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 2)$ | 0 | $\wp + i + 5$ | $i + 3$ | 3 | $\wp + 2 - i$ |
| $a_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ for $\wp \geq 2$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 3$ | $\lceil \frac{\wp}{2} \rceil + 1 + i$ | 3 | $\lfloor \frac{\wp}{2} \rfloor + 4 - i$ |
| $b_i(1 \leq i \leq \wp)$ | 0 | $\wp + 2 - i$ | $\wp + 3 - i$ | $i + 2$ | $i + 1$ |
| $b_{\wp+1}$ | 0 | 2 | 2 | $\wp + 3$ | $\wp + 3$ |
| $b_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ for $\wp \geq 2$ | 0 | $i + 2$ | 2 | $\wp - i + 4$ | $\wp + 1 + i$ |
| $b_3$ for $\wp = 1$ | 0 | 3 | 2 | 3 | 4 |
| $b_4$ for $\wp = 1$ | 0 | 4 | 2 | 2 | 3 |
| $b_5$ for $\wp = 1$ | 0 | 3 | 3 | 2 | 2 |
| $b_5$ for $\wp = 2$ | 0 | 4 | 2 | 4 | 5 |
| $b_6$ for $\wp = 2$ | 0 | 5 | 2 | 3 | 4 |
| $b_{\lceil \frac{n}{2} \rceil+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 2)$ for $\wp \geq 3$ | 0 | $\lceil \frac{\wp}{2} \rceil + i + 2$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 4 - i$ | $\lfloor \frac{n}{2} \rfloor + 2 - i$ |
| $b_{2\wp+3+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil - 2)$ | 0 | $\wp + i + 4$ | $i + 2$ | 2 | $\wp + 1 - i$ |
| $b_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor + 1)$ for $\wp \geq 2$ | 0 | $\lfloor \frac{n}{2} \rfloor - i + 2$ | $\lceil \frac{\wp}{2} \rceil + i$ | 2 | $\lfloor \frac{\wp}{2} \rfloor + 3 - i$ |
| $c_i(1 \leq i \leq \wp)$ | 0 | $\wp + 1 - i$ | $\wp + 2 - i$ | $i + 1$ | $i$ |

| **Distance of vertex** | **from** $\Theta_1$ | **from** $\Theta_2$ | **from** $\Theta_3$ | **from** $\Theta_4$ | **from** $\Theta_5$ |
|---|---|---|---|---|---|
| $c_{\wp+1}$ | 0 | 1 | 1 | $\wp+2$ | $\wp+1$ |
| $c_{\wp+1+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ for $\wp \geq 2$ | 0 | $i+1$ | 1 | $\wp-i+3$ | $\wp+1+i$ |
| $c_3$ for $\wp=1$ | 0 | 2 | 1 | 2 | 3 |
| $c_4$ for $\wp=1$ | 0 | 3 | 1 | 1 | 2 |
| $c_5$ for $\wp=1$ | 0 | 2 | 2 | 1 | 1 |
| $c_5$ for $\wp=2$ | 0 | 3 | 1 | 3 | 4 |
| $c_6$ for $\wp=2$ | 0 | 4 | 1 | 2 | 3 |
| $c_{\lceil \frac{n}{2} \rceil+i}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor+2)$ for $\wp \geq 3$ | 0 | $\lceil \frac{\wp}{2} \rceil+i+1$ | 1 | $\lfloor \frac{\wp}{2} \rfloor+3-i$ | $\lfloor \frac{n}{2} \rfloor+1-i$ |
| $c_{2\wp+3+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil-2)$ | 0 | $\wp+i+3$ | $i+1$ | 1 | $\wp-i$ |
| $c_{n-\lfloor \frac{\wp}{2} \rfloor+i-1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor+1)$ for $\wp \geq 2$ | 0 | $\lfloor \frac{n}{2} \rfloor-i+1$ | $\lceil \frac{\wp}{2} \rceil+i-1$ | 1 | $\lfloor \frac{\wp}{2} \rfloor+2-i$ |
| $d_i(1 \leq i \leq \wp-1)$ | 0 | $\wp-i$ | $\wp+1-i$ | $i+1$ | $i$ |
| $d_\wp$ | 1 | 0 | 1 | $\wp+1$ | $\wp$ |
| $d_{\wp+1}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor+1)$ | 1 | $i$ | 0 | $\wp-i+3$ | $\wp+i$ |
| $d_3$ for $\wp=1$ | 1 | 2 | 0 | 1 | 2 |
| $d_4$ for $\wp=1$ | 1 | 2 | 1 | 0 | 1 |
| $d_5$ for $\wp=1$ | 1 | 1 | 2 | 1 | 0 |
| $d_5$ for $\wp=2$ | 1 | 3 | 0 | 2 | 3 |
| $d_6$ for $\wp=2$ | 1 | 4 | 0 | 1 | 2 |
| $d_{\lfloor \frac{n}{2} \rfloor+i}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil+1)$ | 1 | $\lfloor \frac{\wp}{2} \rfloor+i+1$ | 0 | $\lfloor \frac{\wp}{2} \rfloor+2-i$ | $\lceil \frac{n}{2} \rceil-i$ |
| $d_{2\wp+i+2}(1 \leq i \leq \lfloor \frac{\wp}{2} \rfloor-1)$ | 1 | $\wp+i+2$ | $i$ | 0 | $\wp-i$ |
| $d_{n-\lceil \frac{\wp}{2} \rceil+i-1}(1 \leq i \leq \lceil \frac{\wp}{2} \rceil)$ | 1 | $\lceil \frac{n}{2} \rceil-i$ | $\lfloor \frac{\wp}{2} \rfloor+i-1$ | 0 | $\lceil \frac{\wp}{2} \rceil-i+1$ |
| $d_n$ | 1 | $\wp$ | $\wp$ | 1 | 0 |

Tables 7-9, justifies that $\Theta$ is FTPG of $U_n$, thus by Lemma 1.1 we have, $4 \leq \mathcal{F}(U_n) \leq 5$. □

## 5 Application of Fault-Tolerant Partition Dimension in Flood Relief Networks

In flood-affected regions of Pakistan, efficient and reliable distribution of relief supplies is critical. A relief distribution network can be modeled as a graph where nodes represent delivery points (such as camps, shelters, or villages), and edges represent transport routes between these points. The entire region is divided into multiple relief zones, each served by a central relief hub, analogous to distribution reservoirs in utility networks. The primary objective is to ensure the timely and efficient delivery of aid—such

as food, clean water, and medical supplies—to all affected zones. However, several operational challenges affect this network:

i. If central relief hubs are located far from delivery points, more transportation time and resources are required, increasing the overall operational burden.

ii. As the number of relief zones increases, so does the complexity and time required for coordination and distribution.

To address these challenges, a partitioned distribution model is employed. In this model, a central relief hub is established in each partition zone, enabling a more flexible and efficient operation. Each hub is responsible for a distinct set of delivery points, significantly reducing travel distances and ensuring rapid relief delivery. We consider modeling the relief region as a graph with structure $T_4$. The vertices represent delivery points and vertex representations are organized based on partition sets, as defined in Theorem 2.2. In large-scale disaster operations, such as those during floods, fault-tolerance is a critical requirement. Relief hubs may become inoperative due to road blockages, equipment failures, or communication breakdowns. To ensure uninterrupted service, the network incorporates redundancy by assigning one-third of the nodes in each partition set as backup relief hubs. The representation of vertices of $T_4$, are as follows:

$r(a_1|\Theta) = (0,1,1,2)$, $r(a_2|\Theta) = (1,0,2,2)$,
$r(a_3|\Theta) = (2,0,1,2)$, $r(a_4|\Theta) = (1,1,0,2)$,
$r(b_1|\Theta) = (0,1,2,1)$, $r(b_2|\Theta) = (2,0,2,1)$,
$r(b_3|\Theta) = (2,1,0,1)$, $r(b_4|\Theta) = (1,2,0,1)$,
$r(c_1|\Theta) = (1,2,2,0)$, $r(c_2|\Theta) = (2,1,2,0)$,
$r(c_3|\Theta) = (3,2,1,0)$, $r(c_4|\Theta) = (2,3,1,0)$.

From the unique representation of vertices, it is evident that even if a primary hub fails, the backup hubs within the same partition ensure that aid continues to reach the designated delivery points. This resilience mechanism demonstrates the fault-tolerant partition dimension of the relief network graph. It ensures no affected population is left without assistance, thereby maintaining service continuity and reliability during critical moments of disaster relief.

## 6 Conclusion

In this study, we investigated the fault-tolerant partition dimension, FTPD, of three important classes of convex polytopes: $T_n$, $R_n$, and $U_n$. Given the

complexity of determining exact FTPD values, we focused on establishing sharp upper and lower bounds. Our results demonstrate that for each class, the FTPD lies within the bounds $4 \leq \mathcal{F}(G) \leq 5$, where $G \in \{T_n, R_n, U_n\}$. These findings contribute to the broader understanding of fault tolerance in networks modeled by convex polytopes. The partition-based approach used here reinforces the robustness of such structures in real-world applications like communication, supply chains, and sensor networks. Additionally, we confirmed that FTPD is at least as hard to compute as partition dimension, reflecting its computational complexity. This work also opens avenues for deeper explorations of FTPD in higher-dimensional or irregular polytopes. We conclude by proposing the challenge of determining the exact FTPD for these polytopes as an open problem.

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## Conflict of Interest

The authors declare that they have no conflict of interest.

## Data availability

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